Unit E4

Group actions

Introduction

In this unit you will explore the idea that the elements of a group can 'act' on the elements of a set. You have already met some instances in which this happens: for example, the elements of a symmetric group S_n act on the elements of the set $\{1, 2, \ldots, n\}$. You will see that studying this idea can lead to new insights, both about groups and about the sets on which they act.

You will meet many examples of such 'group actions' in this unit, and study some properties that they all share. You will see how the idea of a group action can help to unify some of the theory covered in the earlier group theory units. Later in the unit you will learn how we can use group actions to help us solve counting problems that involve symmetry, such as the following: How many different cubes are there with each face painted blue, yellow or red, if two such cubes are regarded as the same when one can be rotated to give the other?

1 Group actions

In this first section you will learn what is meant by a group action, and meet a variety of examples.

It can take a while to develop a good intuitive understanding of what a group action is, so do not be concerned if you still feel uncertain about this after working through the first subsection, which includes the definition. The many examples in the subsections that follow will help to clarify the idea.

1.1 What is a group action?

Throughout the group theory units you have met many groups *whose elements are functions from a set to itself* and whose binary operation is function composition. Here are two examples.

- The symmetric group S_3. Each element of S_3 is a permutation of the set $\{1, 2, 3\}$ and so is a function from the set $\{1, 2, 3\}$ to itself.
- The symmetry group $S(\square)$. Each element of $S(\square)$ is a rotation or reflection of the plane $\mathbb{R}^2$ that maps the square to itself and so is a function from the set $\mathbb{R}^2$ to itself.

In mathematics there are many instances where we have a group $(G, \circ)$ and a set X, as illustrated in Figure 1, and the elements of G 'map' the elements of X to elements of the same set in some way. There may be such a mapping effect even if the elements of G *are not actually functions with domain* X. Here are four examples, starting with a familiar one.

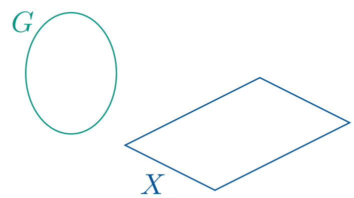


Figure 1 A group $(G, \circ)$ and a set X

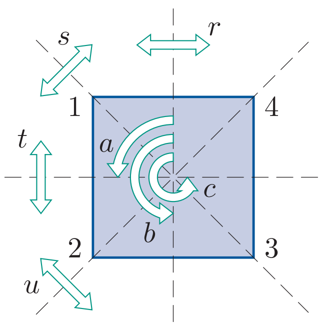


Figure 2 $S(\square)$

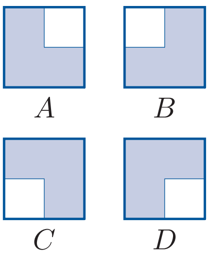


Figure 3 Four modified squares

1. The group $S(\square) = \{e, a, b, c, r, s, t, u\}$ and the set $\{1, 2, 3, 4\}$ of vertex labels of the square, as shown in Figure 2. Each element of $S(\square)$ maps the elements of $\{1, 2, 3, 4\}$ to elements of the same set.
2. The group $S(\square) = \{e, a, b, c, r, s, t, u\}$ and the set $\{A, B, C, D\}$ of modified squares shown in Figure 3. Each element of $S(\square)$ maps the elements of $\{A, B, C, D\}$ to elements of the same set, in the obvious way. For example, the element a of $S(\square)$ maps A to B.
3. The group $\mathrm{GL}(2)$ of all invertible 2×2 matrices with real entries, and the set V, say, of all 2-dimensional column vectors with real entries. One way in which each matrix in $\mathrm{GL}(2)$ can map the vectors in V to vectors in V is by matrix multiplication. For example, the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ maps the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ to the vector
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$
4. The group $(\mathbb{Z}, +)$ and the set $\mathbb{R}$ of real numbers. One way in which each element of $(\mathbb{Z}, +)$ can map the elements of $\mathbb{R}$ to elements of $\mathbb{R}$ is by addition. For example, the element 3 of $(\mathbb{Z}, +)$ adds 3 to each element of $\mathbb{R}$, so it maps 1 to 4, and 0.5 to 3.5, and so on.

In each of these examples there is a 'mapping effect' of the group on the stated set, even though the elements of the group are not functions whose domain is the stated set. In examples 3 and 4 the elements of the group are not functions at all. In examples 1 and 2 the elements of the group *are* functions, but the domain of these functions is not the stated set.

Throughout this unit we will be studying mapping effects like these of groups on sets. Before we can make a start, we need to define a notation that we can use for such mapping effects. When a group element g 'maps' a set element x to another set element, we will not usually denote the 'image' of x 'under' g by $g(x)$, as you might expect. Instead, we will use the notation

$$g \wedge x,$$

which is read as 'g wedge x'. This notation is more convenient in some situations. However, it has the disadvantage that it is less intuitive. Whenever you see it, you might find it helpful to think of it as essentially meaning '$g(x)$'. The group element g is behaving like a function whose domain contains the set element x.

As an example of the notation, consider the effect of the group $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square (see Figure 4). The element a of $S(\square)$ maps vertex 2 to vertex 3, so we write

$$a \wedge 2 = 3.$$

Here is an exercise to help you get used to this notation.

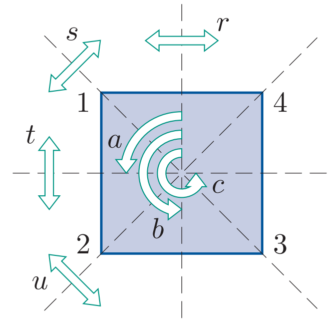


Figure 4 $S(\square)$

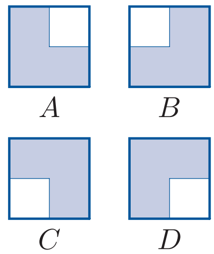


Figure 5 Four modified squares

Exercise E134

(a) Consider the effect of the group $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square, as shown in Figure 4. Write down the following.

(i) $r \wedge 2$ (ii) $b \wedge 1$

(b) Consider the effect of the group $S(\square)$ on the set $\{A, B, C, D\}$ of modified squares shown in Figure 5. Write down the following.

(i) $b \wedge B$ (ii) $s \wedge B$

(c) Consider the effect of the symmetric group S_3 on the set $\{1, 2, 3\}$ of symbols. Write down the following.

(i) $(1\ 3\ 2) \wedge 2$ (ii) $(1\ 2) \wedge 3$

(d) Consider the effect of the group GL(2) of all invertible 2×2 matrices with real entries on the set V of all 2-dimensional column vectors with real entries, by matrix multiplication. Write down the following.

(i) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

(e) Consider the effect of the group $(\mathbb{Z}, +)$ on the set $\mathbb{R}$ of real numbers by addition. Write down the following.

(i) $3 \wedge 7.4$ (ii) $1 \wedge -0.3$

The $g \wedge x$ notation introduced above will be used throughout this unit. Unfortunately, this is just one of several notations used by mathematicians for the 'image' of a set element x 'under' a group element g – there is no standard notation. The alternative notations that you might see in other texts include $g \cdot x$, $g * x$, x^g and simply gx.

We will not be interested in *all* mapping effects of groups on sets. Mathematicians have found that the ones that are useful and interesting are those that have three key properties. These properties are as follows, where $(G, \circ)$ is a group that has a mapping effect on a set X.

- The first property is simply that the elements of G must 'map' the elements of X to elements of the same set X, rather than mapping them to objects that are outside X. That is, for each g in G and each x in X we must have

  $$g \wedge x \in X.$$

  This is illustrated in Figure 6.

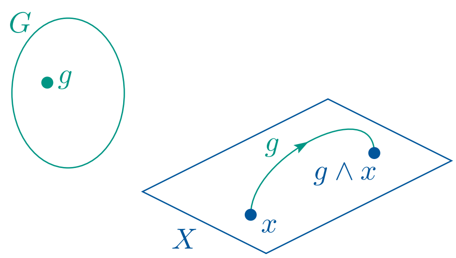


Figure 6 The group element g must 'map' each element of the set X to an element of the set X

- The second property is that the identity element e of $(G, \circ)$ must behave as the *identity function* on X. Recall that the **identity function** on a set X is the function from X to X that **fixes** each element of X, that is, maps each element of X to itself. So, for each x in X we must have

  $$e \wedge x = x.$$

  This is illustrated in Figure 7.

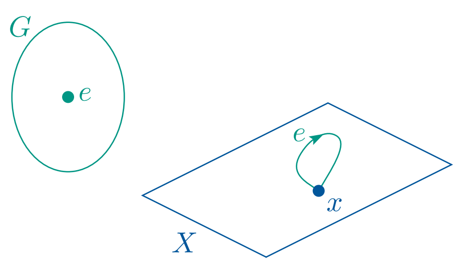


Figure 7 The identity element e of $(G, \circ)$ must 'map' each element x of the set X to itself

- The third property is that the binary operation $\circ$ of the group $(G, \circ)$ must behave like function composition. If the elements of the group $(G, \circ)$ were *actually* functions with domain X and the binary operation $\circ$ were *actually* function composition, then the following equation would be true for all g and h in G and all x in X:

  $$g(h(x)) = (g \circ h)(x).$$

This is just the definition of function composition. So, translating into the $g \wedge x$ notation, we require the binary operation $\circ$ of G to satisfy the following equation for all g and h in G and all x in X:

$$g \wedge (h \wedge x) = (g \circ h) \wedge x.$$

In other words, it must be the case that if g and h are any elements of the group $(G, \circ)$ and x is any element of the set X, then applying h to x and then applying g to the result gives the same answer as applying $g \circ h$ to x. This is illustrated in Figure 8.

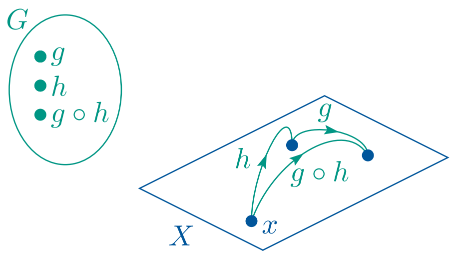


Figure 8 Applying h and then g to x must give the same result as applying $g \circ h$ to x

If a mapping effect of a group $(G, \circ)$ on a set X does have the three properties above, then we say that it is a *group action*, and that $(G, \circ)$ *acts on* X. You will see later in this section that examples 1–4 of mapping effects of groups on sets that were given earlier in this subsection are all group actions.

The definition of a group action is summarised below. Note that the group $(G, \circ)$ and the set X of a group action can each be either finite or infinite.

Definition

Let $(G, \circ)$ be a group with identity element e, and let X be a set. Suppose that for each element g in G and each element x in X an object $g \wedge x$ is defined in some way.

We say that the effect $\wedge$ of $(G, \circ)$ on X is a **group action** of $(G, \circ)$ on X, or simply an **action** of $(G, \circ)$ on X, and that $(G, \circ)$ **acts on** X, if the following three axioms hold.

GA1 Closure For each $g \in G$ and each $x \in X$,

$$g \wedge x \in X.$$

GA2 Identity For each $x \in X$,

$$e \wedge x = x.$$

GA3 Composition For all $g, h \in G$ and all $x \in X$,

$$g \wedge (h \wedge x) = (g \circ h) \wedge x.$$

We call the three axioms in this definition the **group action axioms**.

The next worked exercise demonstrates how to check the group action axioms. In this example, the group is a group of functions, the set on which the group has a mapping effect is the domain of these functions, and $\wedge$ is defined to be simply the usual mapping effect of the functions on the elements of the set, that is, $g \wedge x = g(x)$. (This is in fact the first example of a group of functions given right at the start of this subsection.)

Worked Exercise E55

Consider the symmetric group S_3 and the set $\{1, 2, 3\}$ of symbols. Let $\wedge$ be defined by

$$g \wedge x = g(x)$$

for all $g \in S_3$ and all $x \in \{1, 2, 3\}$. Show that $\wedge$ is a group action.

Solution

We apply the definition of a group action. Here the group $(G, \circ)$ is $(S_3, \circ)$ and the set X is the set $\{1, 2, 3\}$ of symbols.

We check the group action axioms.

GA1 Closure

We have to show that for each $g \in S_3$ and each $x \in \{1, 2, 3\}$,

$$g \wedge x \in \{1, 2, 3\}.$$

Let $g \in S_3$ and let $x \in \{1, 2, 3\}$. Then, since g is a permutation of $\{1, 2, 3\}$,

$$g \wedge x = g(x) \in \{1, 2, 3\}.$$

Thus axiom GA1 holds.

GA2 Identity

We have to show that for each $x \in \{1, 2, 3\}$,

$$e \wedge x = x,$$

where e is the identity element of S_3.

The identity element e of S_3 is the identity permutation of $\{1, 2, 3\}$. So for each $x \in \{1, 2, 3\}$, we have

$$e \wedge x = e(x) = x.$$

Thus axiom GA2 holds.

GA3 Composition

We have to show that for all $g, h \in S_3$ and all $x \in \{1, 2, 3\}$,

$$g \wedge (h \wedge x) = (g \circ h) \wedge x.$$

Let $g, h \in S_3$ and let $x \in \{1, 2, 3\}$. Then

$$\begin{aligned} g \wedge (h \wedge x) &= g \wedge (h(x)) \quad \text{(by the definition of } \wedge) \\ &= g(h(x)) \quad \text{(by the definition of } \wedge) \\ &= (g \circ h)(x) \\ &\qquad\quad \text{(by the definition of function composition)} \\ &= (g \circ h) \wedge x \quad \text{(by the definition of } \wedge). \end{aligned}$$

Thus axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

The reason why axiom GA3 holds in Worked Exercise E55 is, in essence, that when $\wedge$ is defined to be the normal mapping effect of a group of functions on the domain of the functions, the statement of axiom GA3 is just the definition of function composition.

You can practise checking the group action axioms in the next exercise. The group involved is the subgroup of the symmetric group S_5 whose elements are all the permutations in S_5 that either fix the symbols 4 and 5 or transpose them. For example, these permutations include (1 2 3), which fixes 4 and 5, and (1 2 3)(4 5), which transposes 4 and 5. These permutations *do* form a subgroup of S_5, by the result of Exercise E17(a) in Subsection 1.4 of Unit E1 *Cosets and normal subgroups*. You were asked to list the elements of this subgroup in part (b) of that exercise.

Exercise E135

Let G be the subgroup of the symmetric group S_5 that consists of all the permutations in S_5 that either fix the symbols 4 and 5 or transpose them.

Consider this group G and the set $X = \{1, 2, 3\}$. (The set X is a subset of the set $\{1, 2, 3, 4, 5\}$ of symbols permuted by the elements of S_5.) Let $\wedge$ be defined by

$$g \wedge x = g(x)$$

for all $g \in G$ and all $x \in X$. Show that $\wedge$ is a group action.

The next exercise asks you to show that two mapping effects of groups on sets are *not* group actions. Remember that to do this you need only show that *one* of the group action axioms does not hold, by giving a counterexample.

Exercise E136

Show that neither of the following is a group action.

(a) The mapping effect $\wedge$ of the symmetric group S_5 on the set $\{1, 2, 3\}$ of symbols defined by

$$g \wedge x = g(x)$$

for all $g \in S_5$ and all $x \in \{1, 2, 3\}$.

(b) The mapping effect $\wedge$ of the group $(\mathbb{R}^*, \times)$ on the set $\mathbb{R}^2$ of points in the plane defined by

$$g \wedge (x, y) = (x + g, y + g)$$

for all $g \in \mathbb{R}^*$ and all $(x, y) \in \mathbb{R}^2$.

Now consider again the group action in Exercise E135. The group G here is the group of all permutations in S_5 that fix or transpose the symbols 4 and 5, the set X is the set $\{1, 2, 3\}$ of symbols, and the mapping effect $\wedge$ is given by

$$g \wedge x = g(x)$$

for all $g \in G$ and all $x \in X$. (The exercise asked you to show that this *is* a group action.)

This example illustrates an important point about group actions. When a group G acts on a set X, it is possible for two or more different elements of G to have exactly the same mapping effect on the elements of X. That is, two or more different elements of G can 'behave as the same function' from X to X.

For example, for the group action in Exercise E135 the permutations $(1\ 2\ 3)$ and $(1\ 2\ 3)(4\ 5)$ are both elements of the group G, and they both have the same effect on the elements of the set $X = \{1, 2, 3\}$: each of them maps 1 to 2, 2 to 3 and 3 to 1. Similarly, the elements $(1\ 2)$ and $(1\ 2)(4\ 5)$ of G both have the same effect on X, and the elements e and $(4\ 5)$ of G both have the same effect on X, and so on. In fact, for this group action it is possible to pair off the elements of the group G in such a way that the two elements in each pair have the same effect on the set X. (The two elements in each pair are permutations of the form g and $g \circ (4\ 5)$.)

In contrast, for the group action in Worked Exercise E55, which is the usual mapping effect of the symmetric group S_3 on the set $S = \{1, 2, 3\}$ of symbols, all the elements of the group S_3 have *different* effects on the elements of the set S. That is, each group element 'behaves as a different function'.

An action of a group G on a set X in which no two elements of G behave as the same function from X to X is called a **faithful** group action. So the group action in Exercise E135 is not faithful, whereas that in Worked Exercise E55 is faithful.

We now turn to a fundamental property of group actions. To help us describe this property concisely, it is useful to introduce the following definition that applies to functions.

Definition

A one-to-one and onto function from a set X to itself is called a **permutation** of X. (The set X may be either finite or infinite.)

We say that such a function **permutes** the elements of X.

This definition is a generalisation of the similar definition for permutations of *finite* sets, which you met in Subsection 1.1 of Unit B3 *Permutations*. Informally, no matter whether a set is finite or infinite, a permutation of the set is a function that 'shuffles' the elements of the set. For example, reflections and rotations of the plane $\mathbb{R}^2$ are permutations of $\mathbb{R}^2$.

In each of the group actions in Worked Exercise E55 and Exercise E135, the mapping effect of each group element on the set involved in the action is that of a one-to-one and onto function from the set to itself. In other words, each group element behaves as a *permutation* of this set. The theorem below tells us that this is always the case for a group action. Part (a) of the theorem states that the effect of each group element on the set is one-to-one, and part (b) states that it is onto.

Theorem E57

Let $\wedge$ be an action of a group G on a set X. Then $\wedge$ has the following properties.

(a) For each g in G, if x and y are elements of X such that $g \wedge x = g \wedge y$, then $x = y$.

(b) For each g in G, if y is an element of X then there is an element x of X such that $g \wedge x = y$.

Proof

(a) Let g be an element of G and suppose that x and y are elements of X such that $g \wedge x = g \wedge y$. By axiom GA1 the object $g \wedge x$, equal to $g \wedge y$, is an element of X, so $g^{-1} \wedge (g \wedge x)$ and $g^{-1} \wedge (g \wedge y)$ exist and

$$g^{-1} \wedge (g \wedge x) = g^{-1} \wedge (g \wedge y).$$

Hence, by axiom GA3,

$$(g^{-1} \circ g) \wedge x = (g^{-1} \circ g) \wedge y;$$

that is,

$$e \wedge x = e \wedge y.$$

By axiom GA2, this gives

$$x = y,$$

as required.

(b) Let g be an element of G, and let y be an element of X. By axiom GA2,

$$e \wedge y = y,$$

which we can write as

$$(g \circ g^{-1}) \wedge y = y.$$

By axiom GA3, this gives

$$g \wedge (g^{-1} \wedge y) = y.$$

Now $g^{-1} \wedge y$ is an element of X by axiom GA1, so if we take $x = g^{-1} \wedge y$ then $x \in X$ and $g \wedge x = y$, as required. ■

So, by what you have seen in this subsection, we can say the following.

When a group G acts on a set X, we can think of each element of G as behaving like a permutation of the set X, but we have to remember that two or more elements of G may behave like the same permutation.

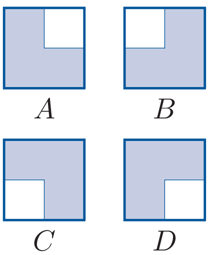


Figure 9 Four modified squares

Often in this unit we will be dealing with a group G that has an *obvious* mapping effect on a set X. For example, the group S_3 has an obvious mapping effect on the set $\{1, 2, 3\}$ of symbols, and the group $S(\square)$ has an obvious mapping effect on the set of modified squares shown in Figure 9, as mentioned earlier. If such a mapping effect is a group action, then we refer to this group action as the **natural action** of G on X, or simply **the action** of G on X. We also say that G **acts** on X **in the natural way**. So where you see a reference to an action of a group G on a set X with no specification of what the action is, you should assume that it is the natural action of G on X. (It is usually possible to define other, less obvious, group actions of the same group G on the same set X.)

We end this subsection with a result that provides many examples of group actions. You saw in Worked Exercise E55 that the usual mapping effect of the group S_3 on the set $\{1, 2, 3\}$ of symbols is a group action. The following more general result holds. Its proof is a straightforward generalisation of the solution to Worked Exercise E55.

Proposition E58

For any natural number n, the usual mapping effect of the group S_n or any of its subgroups on the set $\{1, 2, \ldots, n\}$ of symbols is a group action.

Proof Let n be a natural number, and let $(G, \circ)$ be a subgroup of S_n. The usual mapping effect $\wedge$ of the group $(G, \circ)$ on the set $\{1, 2, \ldots, n\}$ is given by $g \wedge x = g(x)$ for all $g \in G$ and all $x \in \{1, 2, \ldots, n\}$. We check the group action axioms for this mapping effect.

GA1 Closure

Let $g \in G$ and let $x \in \{1, 2, \dots, n\}$. Then, since g is a permutation of $\{1, 2, \dots, n\}$,

$$g \wedge x = g(x) \in \{1, 2, \dots, n\}.$$

Thus axiom GA1 holds.

GA2 Identity

The identity element e of G is the identity permutation of $\{1, 2, \dots, n\}$. So for each $x \in \{1, 2, \dots, n\}$, we have

$$e \wedge x = e(x) = x.$$

Thus axiom GA2 holds.

GA3 Composition

Let $g, h \in G$ and let $x \in \{1, 2, \dots, n\}$. Then

$$\begin{aligned} g \wedge (h \wedge x) &= g \wedge (h(x)) \quad \text{(by the definition of } \wedge\text{)} \\ &= g(h(x)) \quad \text{(by the definition of } \wedge\text{)} \\ &= (g \circ h)(x) \\ &\qquad \text{(by the definition of function composition)} \\ &= (g \circ h) \wedge x \quad \text{(by the definition of } \wedge\text{).} \end{aligned}$$

Thus axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action. ■

Proposition E58 can be generalised still further, to cover groups whose elements are permutations of an *infinite* set, but we will not need this more general result in this unit.

1.2 Actions of groups of symmetries

In this subsection we will look at some examples of group actions in which the group that is acting is the symmetry group of a figure F or one of its subgroups. We call such a group a **group of symmetries** of the figure F.

In most of the examples, the set on which the group acts is a set of figures in $\mathbb{R}^2$ or $\mathbb{R}^3$.

To illustrate the ideas, consider the symmetry group $S(\square)$ (see Figure 10) and the set $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ of modified squares shown in Figure 11. These modified squares are figures in $\mathbb{R}^2$: a figure in $\mathbb{R}^2$ is defined to be a subset of $\mathbb{R}^2$, and each modified square consists of all the points in $\mathbb{R}^2$ that lie on a line or in a shaded area.

Each element of $S(\square)$ has a mapping effect on the eight modified squares, in the obvious way. For example, the element b of $S(\square)$ maps A_1 to A_3.

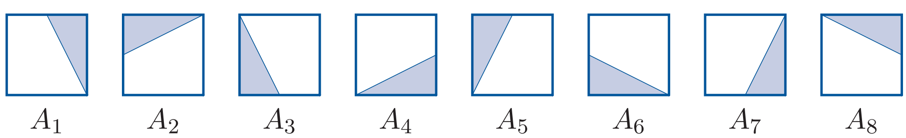


Figure 11 Eight modified squares

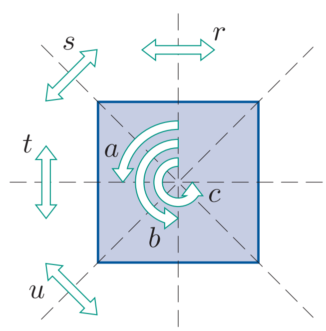


Figure 10 $S(\square)$

This mapping effect of $S(\square)$ on the eight modified squares is a group action, as you will see shortly. To check this, we do not in fact need to check all three group action axioms: we can use Theorem E59 below, which tells us that we need only check axiom GA1, because the other two axioms hold automatically.

The statement of the theorem uses the notation $g(A)$, where A is a figure and g is a transformation, such as a symmetry. This notation means the *image* of A under g, which is given by

$$g(A) = \{g(P) : P \in A\}.$$

In other words, $g(A)$ is the figure obtained by taking the image $g(P)$ of each point P in the figure A under the transformation g. For example, for the figures A_1 and A_3 in Figure 11 (shown again in Figure 12) and the transformation b in $S(\square)$ we have $b(A_1) = A_3$ and $b(A_3) = A_1$.

Figure 12 Two modified squares

Theorem E59

Let G be a group of symmetries of a figure F in $\mathbb{R}^2$, and let X be a set of figures in $\mathbb{R}^2$. Let $\wedge$ be defined by

$$g \wedge A = g(A),$$

for all $g \in G$ and all $A \in X$. Then $\wedge$ is a group action if and only if axiom GA1 (closure) holds.

The same is true if $\mathbb{R}^2$ is replaced by $\mathbb{R}^3$.

Proof To prove the theorem, we have to check that axioms GA2 and GA3 automatically hold in the situation described.

GA2 Identity

Let e be the identity element of G, and let A be a figure in the set X. We have to show that $e \wedge A = A$.

Now e must be the identity transformation. This is because the elements of G are symmetries not only of the figure F but also of the whole plane $\mathbb{R}^2$ (or whole space $\mathbb{R}^3$, as appropriate), so G is a subgroup of the symmetry group of the whole plane $\mathbb{R}^2$ (or space $\mathbb{R}^3$), whose identity element is the identity transformation. Therefore

$$e(P) = P$$

for all points P (in $\mathbb{R}^2$ or $\mathbb{R}^3$ as appropriate). Hence

$$e(A) = A;$$

that is,

$$e \wedge A = A.$$

Thus axiom GA2 holds.

GA3 Composition

Let $g, h \in G$ and let A be a figure in the set X. We have to show that

$$g \wedge (h \wedge A) = (g \circ h) \wedge A.$$

By the definition of $\wedge$, this statement is equivalent to the statement

$$g(h(A)) = (g \circ h)(A).$$

Now, by the definition of function composition, for each point P in the figure A,

$$g(h(P)) = (g \circ h)(P).$$

Hence

$$g(h(A)) = (g \circ h)(A),$$

as required.

Thus axiom GA3 holds.

This completes the proof. ■

The next worked exercise illustrates how to apply Theorem E59.

In the worked exercise, and throughout this unit, you should assume that if an illustrated figure *appears* to have a certain geometric property, then it *does* have that property. For example, in the worked exercise you should assume that each shaded triangle has a vertex that is the midpoint of an edge of the square.

Worked Exercise E56

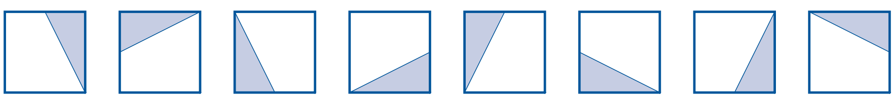
In each of parts (a) and (b) below, let X be the set of modified squares shown, and let $\wedge$ be the mapping effect of the group $S(\square)$ (see Figure 13) on the set X given by

$$g \wedge A = g(A)$$

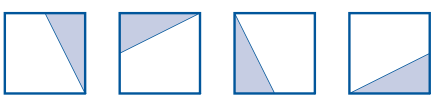
for all $g \in S(\square)$ and all $A \in X$.

In each case, use Theorem E59 to decide whether or not $\wedge$ is a group action. Where it is not a group action, show that it is not.

(a)



(b)



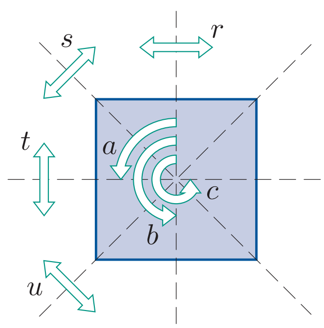


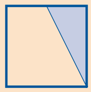
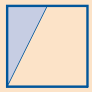
Figure 13 $S(\square)$

Solution

By Theorem E59, in each case $\wedge$ is a group action if and only if axiom GA1 holds, that is, if and only if every element of $S(\square)$ maps each figure in the set X to another figure in X.

(a) We can see by inspection that every symmetry in $S(\square)$ maps each figure in X to another figure in X. So axiom GA1 holds. Hence, by Theorem E59, $\wedge$ is a group action.

(b) The element r of $S(\square)$ maps

 to .

The first figure here is an element of X but the second figure is not. So axiom GA1 does not hold. Hence, by Theorem E59, $\wedge$ is not a group action.

Worked Exercise E56(a) confirms that the effect of $S(\square)$ on the set of eight modified squares in Figure 11 near the start of this subsection is a group action, as claimed.

Exercise E137

In each of parts (a)–(j) below, let $\wedge$ be the mapping effect of the stated group of symmetries on the given set of figures defined by

$$g \wedge A = g(A)$$

for all symmetries g in the group and all figures A in the set of figures.

In each case use Theorem E59 to decide whether or not $\wedge$ is a group action. Where it is not a group action, show that it is not.

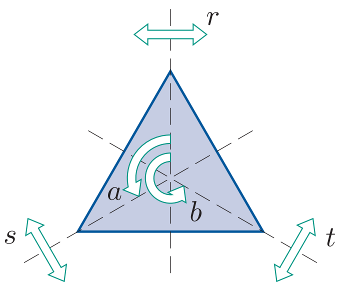


Figure 14 $S(\triangle)$

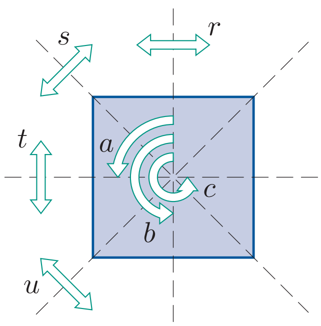
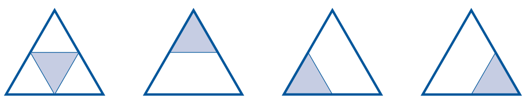
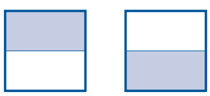


Figure 15 $S(\square)$

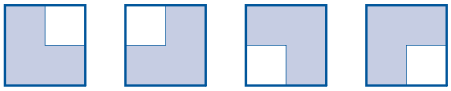
(a) The group $S(\triangle)$ (see Figure 14) and the set X whose elements are the modified triangles shown below.



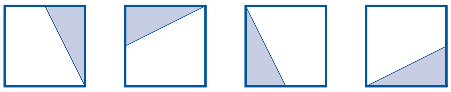
(b) The group $S(\square)$ (see Figure 15) and the set X whose elements are the modified squares shown below.



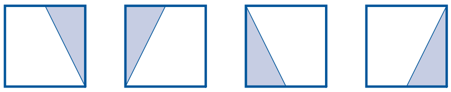
(c) The group $S(\square)$ and the set X whose elements are the modified squares shown below. (These are the modified squares from Figure 3 near the start of the previous subsection.)



(d) The group $S^+(\square)$ of direct symmetries of the square and the set X whose elements are the modified squares shown below.

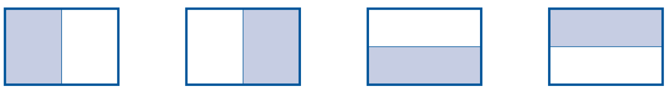


(e) The group $S^+(\square)$ of direct symmetries of the square and the set X whose elements are the modified squares shown below.



(f) The group $S(\square)$ and the set whose elements are the four line segments shown below (the lines of symmetry of the square).

(g) The group $S(\text{▭})$ (see Figure 16) and the set X whose elements are the modified rectangles shown below.



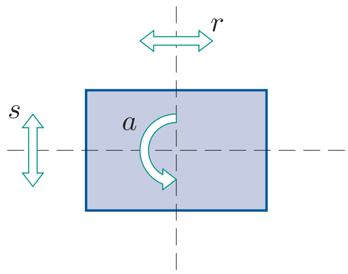
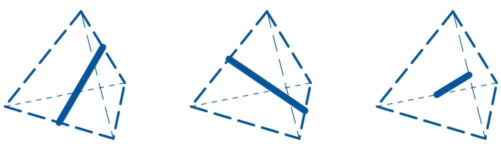


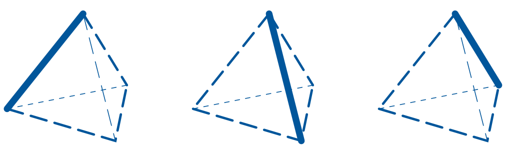
Figure 16 $S(\text{▭})$

(h) The group $S(\square)$ of symmetries of the square and the set X of all plane figures.

(i) The group $S(\text{tet})$ of symmetries of the tetrahedron and the set X whose elements are the three line segments shown below (each line segment joins the midpoints of opposite edges).



(j) The group $S(\text{tet})$ of symmetries of the tetrahedron and the set X whose elements are the three edges shown below.



We can use Theorem E59 to confirm that the mapping effect of $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square is a group action, as claimed in the previous subsection. Each element of $\{1, 2, 3, 4\}$ is a label for a vertex location of the square, and each vertex location of the square is a figure in $\mathbb{R}^2$ that consists of a single point. Every element of $S(\square)$ maps each vertex location of the square to another vertex location of the square, so $\wedge$ is a group action by Theorem E59.

You saw in the previous subsection that when a group G acts on a set X each element of G behaves as a *permutation* of the elements of X. The next worked exercise involves writing down such permutations in cycle notation.

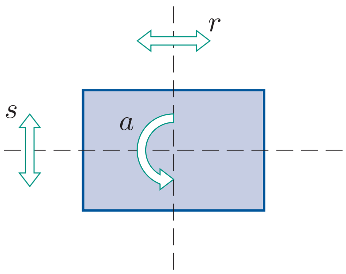
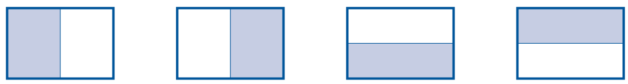


Figure 17 $S(\text{▭})$

Worked Exercise E57

Consider the action of the group $S(\text{▭})$ (see Figure 17) on the set $\{A, B, C, D\}$ whose elements are the modified rectangles shown below.



A B C D

(You saw that this is a group action in Exercise E137(g).)

Write down the effect of each element of $S(\text{▭})$ on the set $\{A, B, C, D\}$ as a permutation in cycle form.

Solution

The permutations are as follows. Here we denote the identity permutation of $\{A, B, C, D\}$ by i, since e is used to denote the identity element of $S(\text{▭})$.

Element g	Permutation
e	i
a	$(A\ B)(C\ D)$
r	$(A\ B)$
s	$(C\ D)$

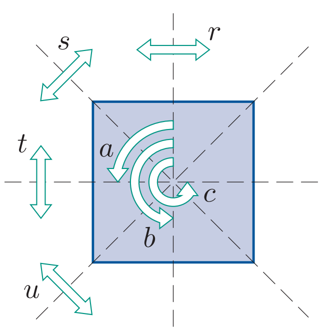


Figure 18 $S(\square)$

Exercise E138

Consider the action of the group $S(\square)$ (see Figure 18) on the set $\{R, S, T, U\}$ of lines of symmetry of the square, as shown below.

R S T U

(You saw that this is a group action in Exercise E137(f).)

Write down the effect of each element of $S(\square)$ on the set $\{R, S, T, U\}$ as a permutation in cycle form.

Exercise E138 illustrates the fact that, as mentioned in Subsection 1.1, when a group G acts on a set X, two or more elements of G may permute the elements of X in the same way. In other words, a group action may not be *faithful*.

You might have observed more in Exercise E138: the group $S(\square)$ splits into pairs such that the group elements in each pair permute the elements of the set $\{R, S, T, U\}$ in the same way. The same is true for the group action in Exercise E135 in Subsection 1.1, as was mentioned in the subsequent text. In fact, whenever a finite group G acts on a set X, the group G can be partitioned into *subsets of equal size* (not necessarily pairs) such that the group elements in each subset permute the elements of X in the same way. If you would like some insight into why this is, then read the optional short section, Section 5, at the end of this unit.

Actions of symmetry groups on sets of coloured figures

So far we have considered the effects of groups of symmetries only on sets of figures, in either $\mathbb{R}^2$ or $\mathbb{R}^3$. However, as you will see in Section 4, sometimes it is useful to consider the effect of a group of symmetries on a slightly different type of set, namely a set of *coloured* figures. As you would expect, a **coloured figure** is a figure each of whose points has been assigned a colour from a finite set of colours.

For example, consider the group $S(\square)$ (see Figure 19) and the set whose elements are the modified squares shown in Figure 20.

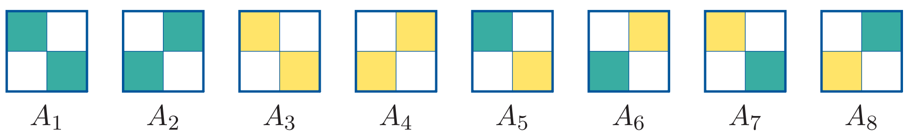


Figure 20 Eight modified squares

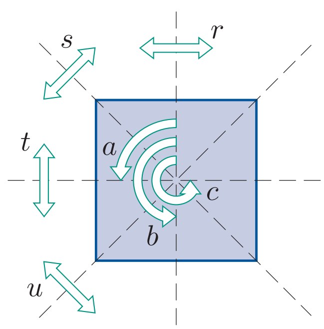


Figure 19 $S(\square)$

Each element of $S(\square)$ has a mapping effect on these eight coloured figures, in the obvious way. For example, the element a of $S(\square)$ maps A_5 to A_6.

Theorem E59 can be generalised to apply to coloured figures, as well as to ordinary figures. To do this, we need to formally define what we mean by the *image* of a coloured figure under an isometry, such as a symmetry. The definition is just as you would expect, as follows. Suppose that g is an isometry of $\mathbb{R}^2$ or $\mathbb{R}^3$ and A is a coloured figure in $\mathbb{R}^2$ or $\mathbb{R}^3$ as appropriate. Then the **image** $g(A)$ of A under g consists of the set of points $\{g(P) : P \in A\}$, with each point $g(P)$ in $g(A)$ assigned the same colour as the corresponding point P in A. For example, for the coloured figures in Figure 20, $a(A_5) = A_6$.

Using this definition, we can generalise Theorem E59 as follows.

Theorem E60

Let G be a group of symmetries of a figure F in $\mathbb{R}^2$, and let X be a set of coloured figures in $\mathbb{R}^2$. Let $\wedge$ be defined by

$$g \wedge A = g(A),$$

for all $g \in G$ and all $A \in X$. Then $\wedge$ is a group action if and only if axiom GA1 (closure) holds.

The same is true if $\mathbb{R}^2$ is replaced by $\mathbb{R}^3$.

Theorem E60 can be proved in the same way as Theorem E59, except that instead of the proof involving each point P in a figure A, it involves each pair (P, c) where P is a point in the figure A and c is the colour assigned to P. The details are omitted here.

Exercise E139

In each of parts (a), (b) and (c) below, let $\wedge$ be the mapping effect of the stated group of symmetries on the given set of coloured figures defined by

$$g \wedge A = g(A)$$

for all symmetries g in the group and all figures A in the set of figures.

In each case use Theorem E60 to decide whether or not $\wedge$ is a group action. Where it is not a group action, show that it is not.

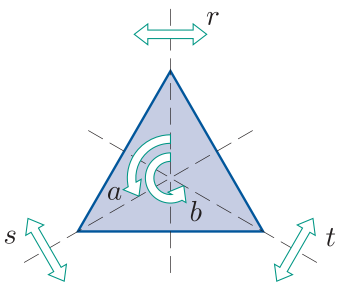


Figure 21 $S(\triangle)$

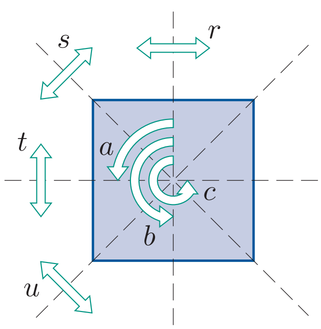
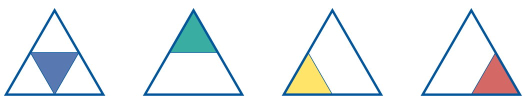
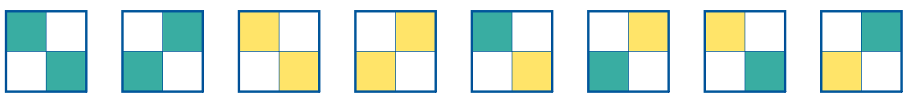


Figure 22 $S(\square)$

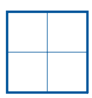
(a) The group $S(\triangle)$ (see Figure 21) and the set X whose elements are the modified triangles shown below.



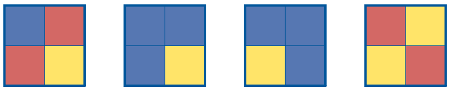
(b) The group $S(\square)$ (see Figure 22) and the set X whose elements are the eight modified squares shown below. (These are the modified squares from Figure 20.)



(c) The group $S(\square)$ and the set X whose elements are all the modified squares obtained by colouring each of the four small squares in the figure below blue, yellow or red.



Some examples of such modified squares are shown below. There are $3^4 = 81$ of them altogether.



Throughout the rest of this unit, we will often work with actions of groups of symmetries on sets of figures or coloured figures. Where you see such an action mentioned, it should be apparent that it *is* a group action by Theorem E59 or Theorem E60. For brevity this is not mentioned every time. (It is mentioned for the first few occurrences.)

1.3 Actions of groups of numbers

In this subsection we will look at some examples of group actions in which the group involved is a group of numbers.

Worked Exercise E58

Consider the group $(\mathbb{Z}, +)$ and the set $\mathbb{R}$. Let $\wedge$ be defined by

$$g \wedge x = g + x$$

for each $g \in \mathbb{Z}$ and each $x \in \mathbb{R}$. Show that $\wedge$ is a group action.

Solution

We check the group action axioms.

GA1 Closure

We have to show that for each $g \in \mathbb{Z}$ and each $x \in \mathbb{R}$,

$$g \wedge x \in \mathbb{R}.$$

Let $g \in \mathbb{Z}$ and let $x \in \mathbb{R}$. Then

$$g \wedge x = g + x \in \mathbb{R}.$$

Thus axiom GA1 holds.

GA2 Identity

The identity element of the group $(\mathbb{Z}, +)$ is 0.

We have to show that for each $x \in \mathbb{R}$,

$$0 \wedge x = x.$$

Let $x \in \mathbb{R}$. Then

$$0 \wedge x = 0 + x = x.$$

Thus axiom GA2 holds.

GA3 Composition

The binary operation of the group $(\mathbb{Z}, +)$ is $+$, so we have to show that for all $g, h \in \mathbb{Z}$ and all $x \in \mathbb{R}$,

$$g \wedge (h \wedge x) = (g + h) \wedge x.$$

Let $g, h \in \mathbb{Z}$ and let $x \in \mathbb{R}$. We have to show that

$$g \wedge (h \wedge x) = (g + h) \wedge x.$$

Now

$$\begin{aligned} g \wedge (h \wedge x) &= g \wedge (h + x) && \text{(by the definition of } \wedge) \\ &= g + (h + x) && \text{(by the definition of } \wedge) \\ &= (g + h) + x \\ &= (g + h) \wedge x && \text{(by the definition of } \wedge). \end{aligned}$$

Thus axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

Exercise E140

Consider the group $(\mathbb{Z}, +)$ and the set $\mathbb{R}$. Let $\wedge$ be defined by

$$g \wedge x = x - g$$

for each $g \in \mathbb{Z}$ and each $x \in \mathbb{R}$. Show that $\wedge$ is a group action.

Exercise E141

Consider the group $(\mathbb{Z}, +)$ and the set $\mathbb{R}$. Let $\wedge$ be defined by

$$g \wedge x = g - x$$

for each $g \in \mathbb{Z}$ and each $x \in \mathbb{R}$. Show that $\wedge$ is not a group action.

In the next worked exercise, the group of real numbers under addition acts on the set of points in the plane.

Worked Exercise E59

Consider the group $(\mathbb{R}, +)$ and the set $\mathbb{R}^2$. Let $\wedge$ be defined by

$$g \wedge (x, y) = (x + yg, y)$$

for all $g \in \mathbb{R}$ and all $(x, y) \in \mathbb{R}^2$.

Show that $\wedge$ is a group action.

Solution

We check the group action axioms.

GA1 Closure

Let $g \in \mathbb{R}$ and let $(x, y) \in \mathbb{R}^2$. Then

$$g \wedge (x, y) = (x + yg, y) \in \mathbb{R}^2.$$

Thus axiom GA1 holds.

GA2 Identity

The identity element of the group $(\mathbb{R}, +)$ is 0.

Let $(x, y) \in \mathbb{R}^2$. Then

$$0 \wedge (x, y) = (x + y \times 0, y) = (x, y).$$

Thus axiom GA2 holds.

GA3 Composition

Let $g, h \in \mathbb{R}$ and let $(x, y) \in \mathbb{R}^2$.

We have to show that

$$g \wedge (h \wedge (x, y)) = (g + h) \wedge (x, y).$$

Now

$$\begin{aligned} g \wedge (h \wedge (x, y)) &= g \wedge (x + yh, y) \quad \text{(by the definition of } \wedge\text{)} \\ &= (x + yh + yg, y) \quad \text{(by the definition of } \wedge\text{)} \end{aligned}$$

and

$$\begin{aligned} (g + h) \wedge (x, y) &= (x + (g + h)y, y) \quad \text{(by the definition of } \wedge\text{)} \\ &= (x + yh + yg, y). \end{aligned}$$

The two expressions obtained are the same, so axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

In Worked Exercise E59 the equation in axiom GA3 was checked by simplifying each side separately and confirming that the same expression is obtained in each case. This can be a helpful approach when the definition of $\wedge$ is complicated.

Exercise E142

Consider the group $(\mathbb{R}, +)$ and the set $\mathbb{R}^2$. Let $\wedge$ be defined by

$$g \wedge (x, y) = (x, y + g)$$

for all $g \in \mathbb{R}$ and all $(x, y) \in \mathbb{R}^2$.

Show that $\wedge$ is a group action.

1.4 Actions of matrix groups

Our final collection of examples of group actions in this section consists of actions of groups of 2×2 matrices on the plane $\mathbb{R}^2$.

First consider the group $\mathrm{GL}(2)$ of invertible 2×2 matrices with real entries under matrix multiplication and the set $\mathbb{R}^2$ of points in the plane. The group $\mathrm{GL}(2)$ has a mapping effect on the set $\mathbb{R}^2$ given by matrix multiplication on the left, as follows: if

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\mathrm{GL}(2)$ and (x, y) is a point in $\mathbb{R}^2$, then the matrix $\mathbf{A}$ maps the point (x, y) to the point $(ax + by, cx + dy)$, because

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

This mapping effect is a group action, as stated and proved below.

Theorem E61

Let $\wedge$ be defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \wedge (x, y) = (ax + by, cx + dy)$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)$ and all points $(x, y) \in \mathbb{R}^2$. Then $\wedge$ is an action of the group $\mathrm{GL}(2)$ on the set $\mathbb{R}^2$.

Proof We check the group action axioms.

GA1 Closure

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)$ and let $(x, y) \in \mathbb{R}^2$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \wedge (x, y) = (ax + by, cx + dy) \in \mathbb{R}^2.$$

Thus axiom GA1 holds.

GA2 Identity

The identity element of the group $\mathrm{GL}(2)$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $(x, y) \in \mathbb{R}^2$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (1x + 0y, 0x + 1y) = (x, y).$$

Thus axiom GA2 holds.

GA3 Composition

Let $\begin{pmatrix} p & q \\ r & s \end{pmatrix}, \begin{pmatrix} t & u \\ v & w \end{pmatrix} \in \mathrm{GL}(2)$ and let $(x, y) \in \mathbb{R}^2$. We have to show that

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \wedge \left(\begin{pmatrix} t & u \\ v & w \end{pmatrix} \wedge (x, y) \right) = \left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} t & u \\ v & w \end{pmatrix} \right) \wedge (x, y).$$

Now

$$\begin{aligned}
&\begin{pmatrix} p & q \\ r & s \end{pmatrix} \wedge \left(\begin{pmatrix} t & u \\ v & w \end{pmatrix} \wedge (x, y) \right) \\
&= \begin{pmatrix} p & q \\ r & s \end{pmatrix} \wedge (tx + uy, vx + wy) \quad \text{(by the definition of } \wedge\text{)} \\
&= (p(tx + uy) + q(vx + wy),\ r(tx + uy) + s(vx + wy)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the definition of } \wedge\text{)} \\
&= (ptx + puy + qvx + qwy,\ rtx + ruy + svx + swy)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\begin{pmatrix} p & q \\ r & s \end{pmatrix} \times \begin{pmatrix} t & u \\ v & w \end{pmatrix} \right) \wedge (x, y) \\
&= \begin{pmatrix} pt + qv & pu + qw \\ rt + sv & ru + sw \end{pmatrix} \wedge (x, y) \\
&= ((pt + qv)x + (pu + qw)y,\ (rt + sv)x + (ru + sw)y) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the definition of } \wedge\text{)} \\
&= (ptx + qvx + puy + qwy,\ rtx + svx + ruy + swy) \\
&= (ptx + puy + qvx + qwy,\ rtx + ruy + svx + swy) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by rearranging the terms).}
\end{aligned}$$

The two expressions obtained are the same, so axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action. ■

Matrix multiplication is just one of many ways in which a group of 2×2 matrices with real entries can act on the plane $\mathbb{R}^2$. We now look at some other examples.

Worked Exercise E60

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}$$

and consider the group $(G, \times)$ and the plane $\mathbb{R}^2$. Let $\wedge$ be defined by

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \wedge (x, y) = (ax, ay)$$

for all $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$. Show that $\wedge$ is a group action.

(You saw that G is a subgroup of GL(2) in Exercise E21(a) in Unit E1, where it was denoted by M.)

Solution

We check the group action axioms.

GA1 Closure

The element (ax, ay) is an element of $\mathbb{R}^2$ for all real numbers a, x and y, so axiom GA1 holds.

GA2 Identity

The identity element of the group G is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $(x, y) \in \mathbb{R}^2$. Then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (1x, 1y) = (x, y)$.

Thus axiom GA2 holds.

GA3 Composition

Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in G$ and let $(x, y) \in \mathbb{R}^2$.

We have to show that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \wedge \left(\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \wedge (x, y) \right) = \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \right) \wedge (x, y).$$

Now

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \wedge \left(\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \wedge (x, y) \right) &= \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \wedge (cx, cy) \\ &= (acx, acy) \end{aligned}$$

and

$$\begin{aligned} \left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \right) \wedge (x, y) &= \begin{pmatrix} ac & ad + bc \\ 0 & ac \end{pmatrix} \wedge (x, y) \\ &= (acx, acy). \end{aligned}$$

The two expressions obtained are the same, so axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

Exercise E143

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}.$$

It is straightforward to show that G is a group under matrix multiplication, and you may assume this. (One of the additional exercises on Section 2 of Unit E1 asks you to show it.)

Determine which of the following, where $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$ and $(x, y) \in \mathbb{R}^2$, define a group action of $(G, \times)$ on $\mathbb{R}^2$.

(a) $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (ax, y)$

(b) $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (ax, by)$

(c) $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (ax, by + y)$

2 Orbits and stabilisers

In this section you will meet the ideas of *orbits* and *stabilisers* for a group action. These ideas will be used throughout the rest of the unit.

2.1 Orbits

To illustrate the idea of an *orbit*, consider the action of the group $S(\square)$ (see Figure 23) on the set $\{R, S, T, U\}$ of lines of symmetry of the square, shown in Figure 24. This is a group action by Theorem E59, as you saw in Exercise E137(f) in Subsection 1.2.

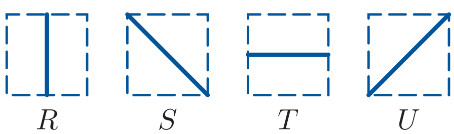


Figure 24 The lines of symmetry of the square

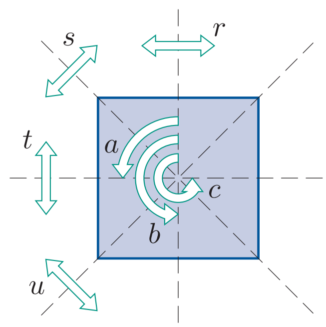


Figure 23 $S(\square)$

Let us choose a particular element of the set $\{R, S, T, U\}$, say R, and consider which elements of the set can be obtained from this element under the action of $S(\square)$.

- If we map R using e, b, r or t, then we obtain R.
- If we map R using a, c, s or u, then we obtain T.

So under the action of $S(\square)$ the element R is mapped to R and T but to no other element. We say that the *orbit* of R under this group action is $\{R, T\}$, and we write $\operatorname{Orb} R = \{R, T\}$.

Here is the general definition of an *orbit*.

Definition

Let $\wedge$ be an action of a group G on a set X, and let x be an element of X. The **orbit** of x under $\wedge$, denoted by $\operatorname{Orb} x$, is

$$\operatorname{Orb} x = \{g \wedge x : g \in G\}.$$

That is, $\operatorname{Orb} x$ is the set of elements of X that can be obtained from x under the action of G.

So, if a group G acts on a set X, then the orbit $\operatorname{Orb} x$ of an element x of X is the subset of X that we obtain if we act on x using each element of G in turn. This is illustrated in Figure 25.

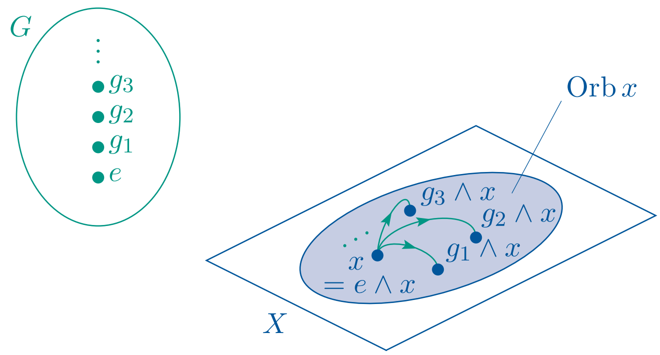


Figure 25 The orbit of an element x

Keep in mind that, as illustrated in Figure 25, $\operatorname{Orb} x$ is a subset of the set X involved in the group action, not a subset of the group G.

As illustrated in Figure 25, the orbit $\operatorname{Orb} x$ of a set element x under a group action always contains the element x itself. This is because $x = e \wedge x$ where e is the identity element of the group G that is acting, by axiom GA2.

In the worked exercise below we find the orbit of another set element under the action of $S(\square)$ on the lines of symmetry of the square.

Worked Exercise E61

Find $\operatorname{Orb} S$ for the action of the group $S(\square)$ (see Figure 26) on the set $\{R, S, T, U\}$ of lines of symmetry of the square (shown on a single diagram in Figure 27).

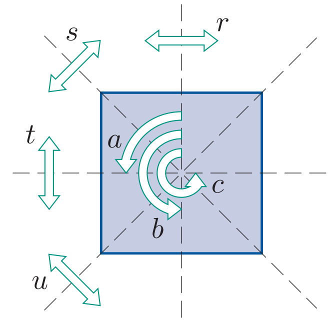


Figure 26 $S(\square)$

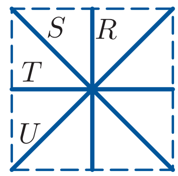


Figure 27 The lines of symmetry of the square

Solution

We have

$$\begin{aligned}\operatorname{Orb} S &= \{g \wedge S : g \in S(\square)\}\\ &= \{e \wedge S,\ a \wedge S,\ b \wedge S,\ c \wedge S,\ r \wedge S,\ s \wedge S,\ t \wedge S,\ u \wedge S\}\\ &= \{S, U, S, U, U, S, U, S\}\\ &= \{S, U\}.\end{aligned}$$

If we are dealing with an action of a *group of symmetries*, then we can usually quickly write down the orbit of a set element just by considering the effects of the symmetries, as demonstrated below.

Worked Exercise E62

Consider again the action of the group $S(\square)$ (see Figure 26) on the set $\{R, S, T, U\}$ of lines of symmetry of the square (see Figure 27). Write down the orbit of each of R, S, T and U.

Solution

There are symmetries of the square that map R to R and to T, but none that map R to S or U. Thus $\operatorname{Orb} R = \{R, T\}$. We can find the orbits of the other lines of symmetry in a similar way.

The orbits are

$$\begin{aligned}\operatorname{Orb} R &= \{R, T\},\\ \operatorname{Orb} S &= \{S, U\},\\ \operatorname{Orb} T &= \{R, T\},\\ \operatorname{Orb} U &= \{S, U\}.\end{aligned}$$

Exercise E144

Consider the action of the group $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square (see Figure 28). Write down the orbit of each vertex label.

(This mapping effect was shown to be a group action immediately following Exercise E137 in Subsection 1.2.)

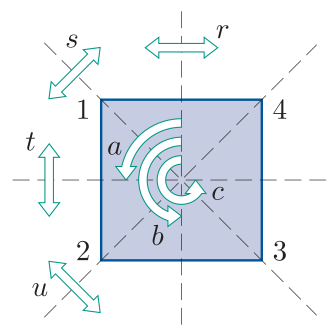


Figure 28 $S(\square)$

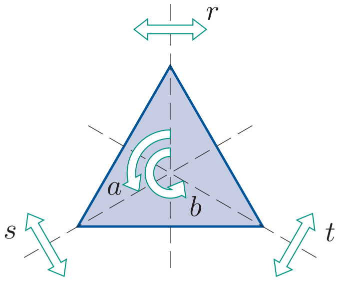


Figure 29 $S(\triangle)$

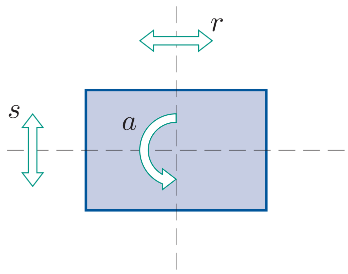
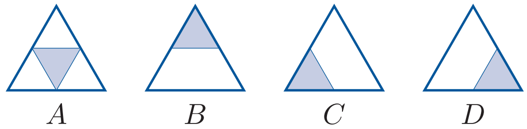


Figure 30 $S(\square)$

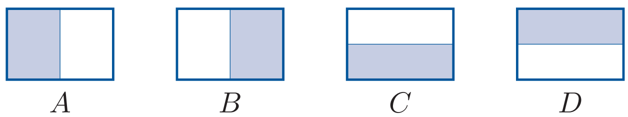
Exercise E145

Consider the action of the group $S(\triangle)$ (see Figure 29) on the set $\{A, B, C, D\}$ of modified triangles shown below. Write down the orbit of each of A, B, C and D.



Exercise E146

Consider the action of the group $S(\square)$ (see Figure 30) on the set $\{A, B, C, D\}$ of modified rectangles shown below. Write down the orbit of each of A, B, C and D.



In the next worked exercise we find the orbits of set elements under the action of an infinite group.

Worked Exercise E63

Consider the action of the group $S^+(\bigcirc)$ of direct symmetries of the disc on the plane $\mathbb{R}^2$, assuming that the disc is placed with its centre at the origin O. Describe geometrically the orbit of each point in $\mathbb{R}^2$.

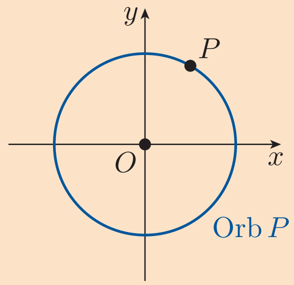
Solution

The group $S^+(\bigcirc)$ is the group of rotations about the origin O.

Any rotation about O maps O to itself, so

$$\operatorname{Orb} O = \{O\}.$$

Now let P be any other point in $\mathbb{R}^2$. The elements of $S^+(\bigcirc)$ rotate P about O, through all possible angles. So $\operatorname{Orb} P$ is the circle with centre O whose radius is the distance between O and P, as shown below.



Exercise E147

Consider the action of the group $S(\bigcirc)$ of *all* symmetries of the disc on the plane $\mathbb{R}^2$, assuming that the disc is placed with its centre at the origin O. Describe geometrically the orbit of each point in $\mathbb{R}^2$.

(Remember that the elements of $S(\bigcirc)$ are the rotations about O and the reflections in the lines through O.)

You may have noticed that in all the examples of orbits that you have met so far, the orbits of any two different elements of the set involved are always either *identical sets* or *disjoint sets*: they never partially overlap. For example, consider the orbits of the lines of symmetry R, S, T and U of the square under the action of the group $S(\square)$, which were found in Worked Exercise E62:

$$\begin{aligned} \operatorname{Orb} R &= \{R, T\}, \\ \operatorname{Orb} S &= \{S, U\}, \\ \operatorname{Orb} T &= \{R, T\}, \\ \operatorname{Orb} U &= \{S, U\}. \end{aligned}$$

Here, for instance, $\operatorname{Orb} R$ and $\operatorname{Orb} T$ are the same set, while $\operatorname{Orb} R$ and $\operatorname{Orb} S$ are disjoint sets.

Also, under any group action, every element of the set involved lies in some orbit. This is because every element of the set lies in its own orbit, at least.

So, in all the examples that you have met, the distinct orbits form a *partition* of the set on which the group acts. For example, the distinct orbits under the action of the group $S(\square)$ on the set $\{R, S, T, U\}$ of lines of symmetry of the square form the following partition of the set $\{R, S, T, U\}$:

$$\{R, T\}, \quad \{S, U\}.$$

The distinct orbits under a group action always form a partition of the set on which the group acts, as proved below.

Theorem E62

Let $\wedge$ be an action of a group $(G, \circ)$ on a set X. Then the distinct orbits of the elements of X under $\wedge$ form a partition of X.

Proof To prove the theorem, we define a relation $\sim$ on X, prove that $\sim$ is an equivalence relation, and show that its equivalence classes are the distinct orbits under $\wedge$ of the elements of X. (There is a reminder of the definition of an equivalence relation in Subsection 4.1 of Unit E1.)

Let the relation $\sim$ be defined on X by

$$x \sim y \quad \text{if } y \in \operatorname{Orb} x.$$

To prove that $\sim$ is an equivalence relation, we show that $\sim$ has the reflexive, symmetric and transitive properties. Let the identity element of $(G, \circ)$ be e.

E1 Reflexivity

Let $x \in X$. We have to show that $x \sim x$. That is, we have to show that

$$x \in \operatorname{Orb} x.$$

This is true (as mentioned earlier), because $x = e \wedge x$. So $x \sim x$. Thus $\sim$ is reflexive.

E2 Symmetry

Let $x, y \in X$, and suppose that $x \sim y$. Then

$$y \in \operatorname{Orb} x.$$

We have to show that $y \sim x$, that is, $x \in \operatorname{Orb} y$.

Since $y \in \operatorname{Orb} x$, there is an element g in G such that

$$y = g \wedge x.$$

Hence

$$\begin{aligned} g^{-1} \wedge y &= g^{-1} \wedge (g \wedge x) \\ &= (g^{-1} \circ g) \wedge x \quad \text{(by axiom GA3)} \\ &= e \wedge x \\ &= x \quad \text{(by axiom GA2).} \end{aligned}$$

Since $x = g^{-1} \wedge y$ and $g^{-1} \in G$, we have $x \in \operatorname{Orb} y$, and hence $y \sim x$. Thus $\sim$ is symmetric.

GA3 Transitivity

Let $x, y, z \in X$, and suppose that $x \sim y$ and $y \sim z$; that is,

$$y \in \operatorname{Orb} x \quad \text{and} \quad z \in \operatorname{Orb} y.$$

We have to show that $x \sim z$, that is, $z \in \operatorname{Orb} x$.

Since $y \in \operatorname{Orb} x$ and $z \in \operatorname{Orb} y$, there are elements g and h in G such that

$$y = g \wedge x \quad \text{and} \quad z = h \wedge y.$$

Hence

$$\begin{aligned} z &= h \wedge y \\ &= h \wedge (g \wedge x) \quad (\text{since } y = g \wedge x) \\ &= (h \circ g) \wedge x \quad (\text{by axiom GA3}). \end{aligned}$$

Since $z = (h \circ g) \wedge x$ and $h \circ g \in G$, we have $z \in \operatorname{Orb} x$, that is, $x \sim z$. Thus $\sim$ is transitive.

Hence $\sim$ has the reflexive, symmetric and transitive properties, so it is an equivalence relation on X.

The equivalence classes of any equivalence relation form a partition of the set on which the relation is defined (by Theorem A16, restated in Subsection 4.1 of Unit E1). Hence the equivalence classes of $\sim$ form a partition of X. But the equivalence class of any element x of X is

$$\{y \in X : x \sim y\} = \{y \in X : y \in \operatorname{Orb} x\} = \operatorname{Orb} x.$$

Thus the equivalence classes are precisely the distinct orbits under $\wedge$, so the orbits form a partition of X. ■

The part of the proof of Theorem E62 that deals with the symmetric property shows that for any group action $\wedge$,

$$\text{if } y = g \wedge x, \text{ then } x = g^{-1} \wedge y.$$

This is a useful result that is worth remembering.

We refer to the distinct orbits of elements under a group action as the *orbits of the group action*. Theorem E62 gives us the following strategy for finding the orbits of a group action on a finite set.

Strategy E7

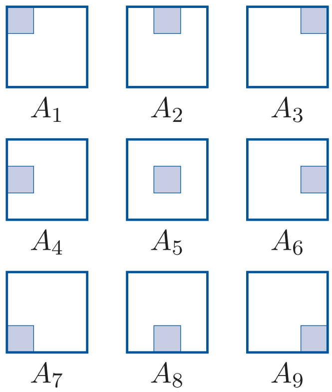
To find the orbits of an action of a group G on a finite set X, do the following.

1. Choose any element x of X, and find its orbit.
2. Choose any element of X not yet assigned to an orbit, and find its orbit.
3. Repeat step 2 until X is partitioned.

We can also use Strategy E7 to find the orbits of an action of a group G on an *infinite* set X when there are only finitely many orbits.

Exercise E148

Consider the action of the group $S(\square)$ (see Figure 31) on the set $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$ of modified squares shown below. Write down the orbits of this group action.



(This is a group action by Theorem E59.)

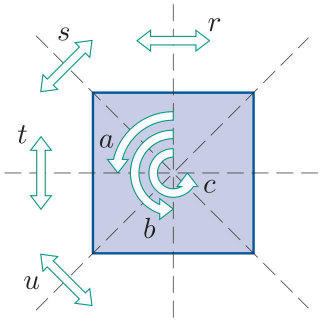
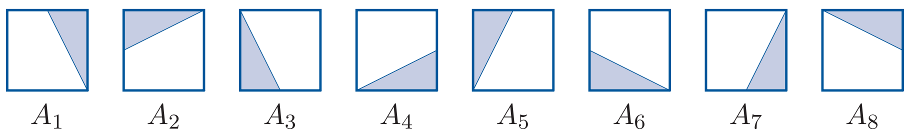


Figure 31 $S(\square)$

Exercise E149

(a) Consider the action of the group $S^+(\square)$ of direct symmetries of the square on the set $X = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ of modified squares shown below. Write down the orbits of this group action.



(b) Now consider the action of the group $S(\square)$ of *all* symmetries of the square on the same set X as in part (a). Write down the orbits of this group action.

(These are group actions by Theorem E59.)

Exercise E150

Consider the action of each of the following subgroups of $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square (see Figure 32). For the action of each subgroup, write down the orbits.

(Each subgroup gives a group action by Theorem E59.)

(a) $S^+(\square)$ (b) $\{e, b, s, u\}$ (c) $\{e, r\}$ (d) $\{e\}$

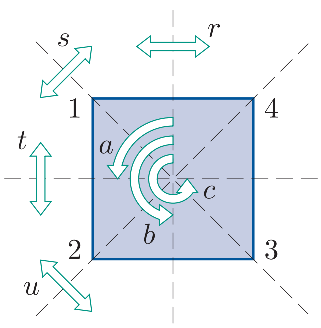


Figure 32 $S(\square)$

2.2 Orbits of group actions on $\mathbb{R}^2$

Worked Exercise E63 in the previous subsection involved the action of a group on the plane $\mathbb{R}^2$. Because the set involved was the plane $\mathbb{R}^2$, we were able to give a geometric description of the orbits of the action: they were the origin and the circles whose centre is the origin. These orbits partition the plane $\mathbb{R}^2$, as we know they must by Theorem E62: some of them are shown in Figure 33.

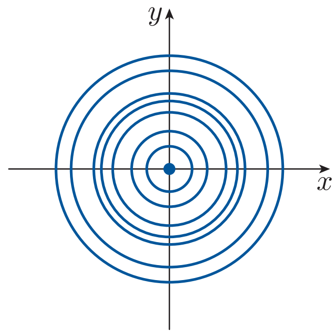


Figure 33 Some of the orbits of the group action in Worked Exercise E63

In this subsection we will find the orbits of some more actions of groups on the plane $\mathbb{R}^2$. Many of the groups involved are matrix groups.

In the first worked exercise we find the orbits of some individual points in $\mathbb{R}^2$ under a group action.

Worked Exercise E64

Let

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}^+ \right\}.$$

Consider the action $\wedge$ of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \wedge (x, y) = (ax, by)$$

for all $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$. (You may assume that $(G, \times)$ is a group and that $\wedge$ is a group action; both of these are straightforward to show.)

Find the orbit of each of the following points in $\mathbb{R}^2$. Describe each of these orbits geometrically.

(a) $(0, 0)$ (b) $(-1, 0)$ (c) $(1, -1)$

Solution

First we find an expression for the orbit of a general point (x, y) in $\mathbb{R}^2$ under this group action.

We have to apply the general definition of an orbit,

$$\operatorname{Orb} x = \{g \wedge x : g \in G\},$$

to the situation here. We

- replace x by a general element of the set $\mathbb{R}^2$, say (x, y)
- replace g by a general element of the group G, say $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

For any point $(x,y) \in \mathbb{R}^2$,

$$\operatorname{Orb}(x,y) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \wedge (x,y) : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G \right\}$$

Now we use the definition of $\wedge$ to simplify the expression in front of the colon. We also simplify the condition after the colon: what it tells us about the values taken by a and b is simply that $a, b \in \mathbb{R}^+$.

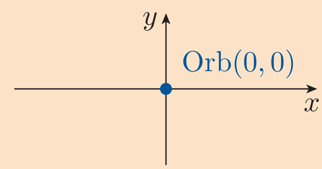
$$= \{(ax, by) : a, b \in \mathbb{R}^+\}.$$

We now have an expression for the orbit of a general point (x,y). We use it to find the orbits of the given points.

(a) Putting $(x,y) = (0,0)$ gives

$$\begin{aligned} \operatorname{Orb}(0,0) &= \{(a \times 0, b \times 0) : a, b \in \mathbb{R}^+\} \\ &= \{(0,0) : a, b \in \mathbb{R}^+\} \\ &= \{(0,0)\}. \end{aligned}$$

So $\operatorname{Orb}(0,0)$ consists of the origin alone.

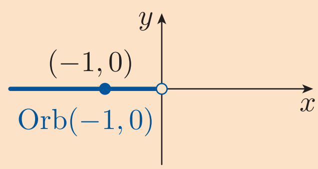


(b) Putting $(x,y) = (-1,0)$ gives

$$\begin{aligned} \operatorname{Orb}(-1,0) &= \{(a \times (-1), b \times 0) : a, b \in \mathbb{R}^+\} \\ &= \{(-a, 0) : a \in \mathbb{R}^+\}. \end{aligned}$$

As a runs through all the values in $\mathbb{R}^+$, the point $(-a, 0)$ moves through all the points on the negative part of the x-axis.

So $\operatorname{Orb}(-1,0)$ is the negative part of the x-axis.

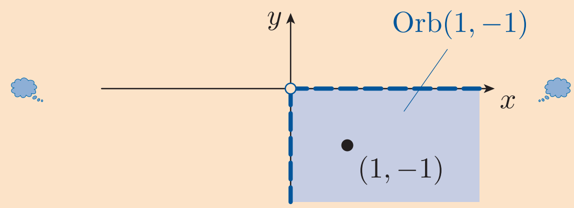


(c) Putting $(x,y) = (1,-1)$ gives

$$\begin{aligned} \operatorname{Orb}(1,-1) &= \{(a \times 1, b \times (-1)) : a, b \in \mathbb{R}^+\} \\ &= \{(a, -b) : a, b \in \mathbb{R}^+\}. \end{aligned}$$

As a and b run through all the values in $\mathbb{R}^+$, the point $(a, -b)$ moves through all the points in the fourth quadrant of the plane.

So Orb$(1,-1)$ is the fourth quadrant of the plane. (It does not include any points on the x-axis or y-axis.)



Exercise E151

For the group action in Worked Exercise E64, find the orbit of each of the following points. Describe each of these orbits geometrically.

(a) $(1,0)$ (b) $(0,-1)$ (c) $(1,1)$

We have now found six of the orbits of the group action in Worked Exercise E64 and Exercise E151. These orbits are illustrated in Figure 34.

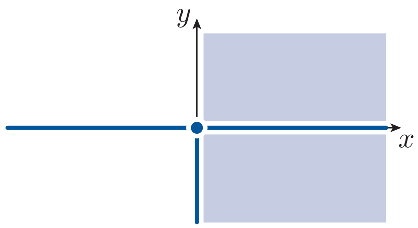


Figure 34 Six of the orbits of the group action in Worked Exercise E64 and Exercise E151

Exercise E152

Find the remaining orbits of the group action in Worked Exercise E64 and Exercise E151, remembering that the orbits partition the plane. Sketch a diagram to show how the orbits partition the plane.

The group action considered in Worked Exercise E64 and the two subsequent exercises has only finitely many orbits (nine altogether). In contrast, the group action in the next worked exercise has infinitely many.

Finding all the orbits of a group action can be quite complicated. Usually it is best to start by finding an expression for the orbit of a general element of the set on which the group acts, as was done in Worked Exercise E64. Next, it is often helpful to use this expression to find the

orbits of a few particular elements of the set, to try to get an idea of how the set might split up into orbits. Finally, you can attempt to confirm what you think happens by using algebraic or geometric arguments. Keep in mind that the orbits partition the set on which the group acts, so to find more orbits you need to consider elements that do not lie in the orbits that you have already found.

Worked Exercise E65

Consider the action $\wedge$ of the group $(\mathbb{R}, +)$ on the set $\mathbb{R}^2$ defined by

$$g \wedge (x, y) = (x + yg, y)$$

for all $g \in \mathbb{R}$ and all $(x, y) \in \mathbb{R}^2$.

(You saw that this is a group action in Worked Exercise E59 in Subsection 1.3.)

Find all the orbits of this group action. Describe them geometrically, and sketch a diagram to show how they partition the plane.

Solution

We start by finding an expression for the orbit of a general point (x, y) under this group action.

For any point $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}\mathrm{Orb}(x, y) &= \{g \wedge (x, y) : g \in \mathbb{R}\}\\ &= \{(x + yg, y) : g \in \mathbb{R}\}.\end{aligned}$$

Let us use this equation to find the orbits of some points, to try to get an idea of what might happen in general. We choose 'simple' points to start with. We have, for example,

$$\begin{aligned}\mathrm{Orb}(0, 0) &= \{(0 + 0g, 0) : g \in \mathbb{R}\} = \{(0, 0) : g \in \mathbb{R}\} = \{(0, 0)\},\\ \mathrm{Orb}(1, 0) &= \{(1 + 0g, 0) : g \in \mathbb{R}\} = \{(1, 0) : g \in \mathbb{R}\} = \{(1, 0)\}.\end{aligned}$$

The working here leads us to realise that any point of the form $(x, 0)$ has an orbit that contains only the point itself. We confirm this formally.

For any point $(x, 0)$, that is, any point on the x-axis, we have

$$\mathrm{Orb}(x, 0) = \{(x + 0g, 0) : g \in \mathbb{R}\} = \{(x, 0) : g \in \mathbb{R}\} = \{(x, 0)\}.$$

Now let us try some other points. We have, for example,

$$\begin{aligned}\mathrm{Orb}(0, 1) &= \{(0 + 1g, 1) : g \in \mathbb{R}\} = \{(g, 1) : g \in \mathbb{R}\},\\ \mathrm{Orb}(0, 2) &= \{(0 + 2g, 2) : g \in \mathbb{R}\} = \{(2g, 2) : g \in \mathbb{R}\}.\end{aligned}$$

As g runs through all values in $\mathbb{R}$, the point $(g, 1)$ moves through all the points on the horizontal line with y-intercept 1. So $\mathrm{Orb}(0, 1)$ is the horizontal line through $(0, 1)$. Similarly, as g runs through all

values in $\mathbb{R}$, the point $(2g, 2)$ moves through all the points on the horizontal line with y-intercept 2. So $\mathrm{Orb}(0, 2)$ is the horizontal line through $(0, 2)$. It looks as if the orbit of any point of the form $(0, y)$, where $y \neq 0$, is the horizontal line through that point. We now check this formally.

For any point $(0, y)$ with $y \neq 0$, that is, any point on the y-axis except the origin, we have

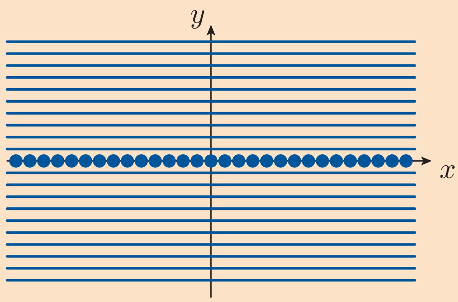
$$\mathrm{Orb}(0, y) = \{(0 + yg, y) : g \in \mathbb{R}\} = \{(yg, y) : g \in \mathbb{R}\}.$$

This is the set of points that lie on the horizontal line through $(0, y)$. So $\mathrm{Orb}(0, y)$, where $y \neq 0$, is the horizontal line through $(0, y)$.

We have now found all the orbits.

Every point in $\mathbb{R}^2$ is either a point on the x-axis or a point on a horizontal line through some point of the form $(0, y)$, where $y \neq 0$. So we have now partitioned the plane $\mathbb{R}^2$ into orbits.

The orbits are the individual points on the x-axis, together with all the horizontal lines other than the x-axis, as sketched below.



Exercise E153

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}.$$

Consider the action $\wedge$ of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (ax, y)$$

for all $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$. (You saw that this is a group action in Exercise E143(a) in Subsection 1.4.)

Find all the orbits of this group action. Describe them geometrically, and sketch a diagram to show how they partition the plane.

Exercise E154

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}.$$

Consider the action $\wedge$ of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \wedge (x, y) = (ax, ay)$$

for all $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$. (You saw that this is a group action in Worked Exercise E60 in Subsection 1.4.)

Find all the orbits of this group action. Describe them geometrically, and sketch a diagram to show how they partition the plane.

2.3 Stabilisers

This subsection introduces the idea of a *stabiliser*.

Definition

Let $\wedge$ be an action of a group G on a set X, and let x be an element of X. The **stabiliser** of x under $\wedge$, denoted by $\operatorname{Stab} x$, is given by

$$\operatorname{Stab} x = \{g \in G : g \wedge x = x\}.$$

That is, $\operatorname{Stab} x$ is the set of elements of G that fix x.

In other words, if a group G acts on a set X and x is an element of X, then the stabiliser $\operatorname{Stab} x$ of x is the set of elements of G that map x to itself. This is illustrated in Figure 35.

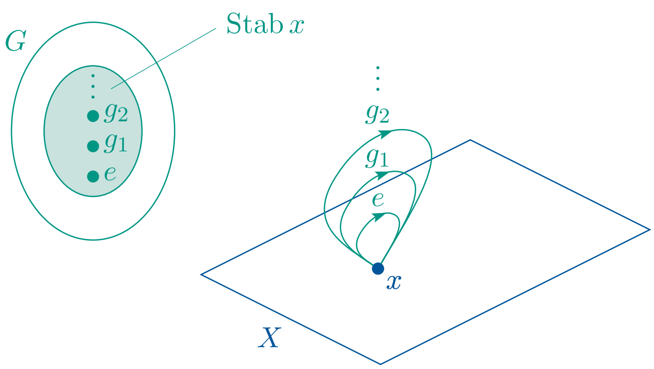


Figure 35 The stabiliser of an element x

Notice that, whereas the orbit of an element x is a subset of the set X, the stabiliser of x is a subset of the group G.

As illustrated in Figure 35, the stabiliser $\operatorname{Stab} x$ of a set element x under the action of a group always contains the identity element e of the group. This is because $e \wedge x = x$, by axiom GA2.

Worked Exercise E66

Find Stab R for the action of the group $S(\square)$ (see Figure 36) on the set $\{R, S, T, U\}$ of lines of symmetry of the square (see Figure 37).

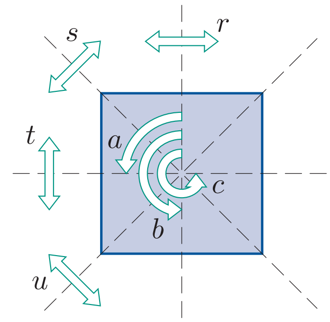


Figure 36 $S(\square)$

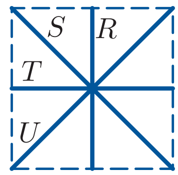


Figure 37 The lines of symmetry of the square

Solution

We have

$$\begin{aligned}
e \wedge R &= R,\\
a \wedge R &= T,\\
b \wedge R &= R,\\
c \wedge R &= T,\\
r \wedge R &= R,\\
s \wedge R &= T,\\
t \wedge R &= R,\\
u \wedge R &= T.
\end{aligned}$$

The elements of $S(\square)$ that fix R are e, b, r and t.

Hence

$$\operatorname{Stab} R = \{e, b, r, t\}.$$

As with orbits, if we are dealing with the action of a group of symmetries, then we can usually quickly write down the stabilisers of set elements just by considering the effects of the symmetries.

Worked Exercise E67

Consider the action of the group $S(\square)$ (see Figure 36) on the set $\{R, S, T, U\}$ of lines of symmetry of the square (see Figure 37). Write down the stabiliser of each of R, S, T and U.

Solution

The symmetries e, b, r and t all map R to itself, but none of the other symmetries do. Hence $\operatorname{Stab} R = \{e, b, r, t\}$. The stabilisers of the other lines of symmetry can be found in a similar way.

The stabilisers are

$$\begin{aligned}
\operatorname{Stab} R &= \{e, b, r, t\},\\
\operatorname{Stab} S &= \{e, b, s, u\},\\
\operatorname{Stab} T &= \{e, b, r, t\},\\
\operatorname{Stab} U &= \{e, b, s, u\}.
\end{aligned}$$

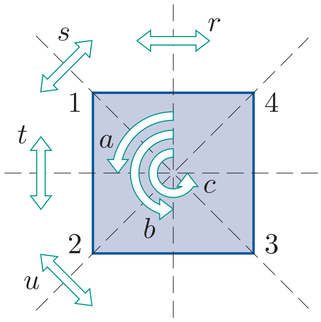


Figure 38 $S(\square)$

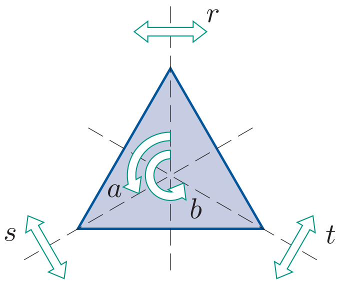


Figure 39 $S(\triangle)$

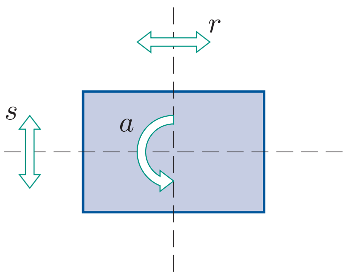


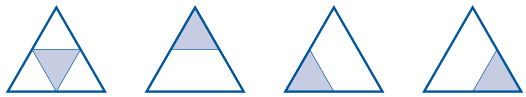
Figure 40 $S(\Box)$

Exercise E155

Consider the action of the group $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square (see Figure 38). Write down the stabiliser of each vertex label.

Exercise E156

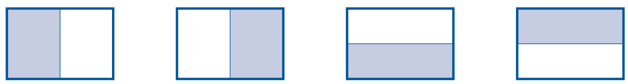
Consider the action of the group $S(\triangle)$ (see Figure 39) on the set $\{A, B, C, D\}$ of modified triangles shown below. Write down the stabiliser of each of A, B, C and D.



A B C D

Exercise E157

Consider the action of the group $S(\Box)$ (see Figure 40) on the set $\{A, B, C, D\}$ of modified rectangles shown below. Write down the stabiliser of each of A, B, C and D.



A B C D

The next worked exercise involves an action of an infinite group.

Worked Exercise E68

Consider the action of the group $S^+(\bigcirc)$ of direct symmetries of the disc on the plane $\mathbb{R}^2$, assuming that the disc is placed with its centre at the origin O. Find the stabiliser of each point in $\mathbb{R}^2$.

Solution

The group $S^+(\bigcirc)$ is the group of rotations about the origin O.

Any rotation about O fixes O, so $\operatorname{Stab} O$ is the whole of $S^+(\bigcirc)$.

Now let P be any other point in $\mathbb{R}^2$. The only rotation in $S^+(\bigcirc)$ that fixes P is the identity symmetry e (also denoted by r_0). So

$$\operatorname{Stab} P = \{e\}.$$

Exercise E158

Consider the action of the group $S(\bigcirc)$ of *all* symmetries of the disc on the plane $\mathbb{R}^2$, assuming that the disc is placed with its centre at the origin O. Find the stabiliser of each point in $\mathbb{R}^2$.

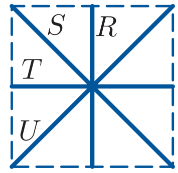


Figure 41 The lines of symmetry of the square

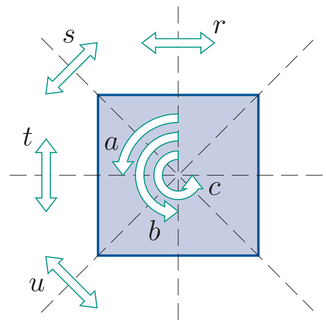


Figure 42 $S(\square)$

You might have noticed that every example of a stabiliser that you have met so far has turned out to be a *subgroup* of the group that is acting, rather than just a subset of the group. For example, consider the stabilisers of the lines of symmetry R, S, T and U of the square (see Figure 41) under the action of the group $S(\square)$ (see Figure 42), which were found in Worked Exercise E67:

$$\begin{aligned}
&\operatorname{Stab} R = \{e, b, r, t\},\\
&\operatorname{Stab} S = \{e, b, s, u\},\\
&\operatorname{Stab} T = \{e, b, r, t\},\\
&\operatorname{Stab} U = \{e, b, s, u\}.
\end{aligned}$$

Both $\{e, b, r, t\}$ and $\{e, b, s, u\}$ are subgroups of $S(\square)$, as you saw in Exercise E15 in Subsection 1.4 of Unit E1.

The stabiliser of a set element under the action of a group is always a subgroup of the group that is acting, as proved below.

Theorem E63

Let $\wedge$ be an action of a group $(G, \circ)$ on a set X. Then, for each element x of X, the set $\operatorname{Stab} x$ is a subgroup of $(G, \circ)$.

Proof Let x be an element of X. We show that the three subgroup properties hold for $\operatorname{Stab} x$.

SG1 Closure

Let $g, h \in \operatorname{Stab} x$. Then

$$g \wedge x = x \quad \text{and} \quad h \wedge x = x.$$

We have

$$\begin{aligned}
(g \circ h) \wedge x &= g \wedge (h \wedge x) \quad \text{(by axiom GA3)}\\
&= g \wedge x\\
&= x.
\end{aligned}$$

Hence $g \circ h \in \operatorname{Stab} x$. Thus property SG1 holds.

SG2 Identity

Let e be the identity element of $(G, \circ)$. Since $e \wedge x = x$, by axiom GA2, it follows that $e \in \operatorname{Stab} x$. Thus property SG2 holds.

SG3 Inverses

Let $g \in \operatorname{Stab} x$. Then

$$g \wedge x = x.$$

It follows that

$$\begin{aligned} g^{-1} \wedge x &= g^{-1} \wedge (g \wedge x) \\ &= (g^{-1} \circ g) \wedge x \quad \text{(by axiom GA3)} \\ &= e \wedge x \\ &= x \quad \text{(by axiom GA2).} \end{aligned}$$

Hence $g^{-1} \in \operatorname{Stab} x$. Thus property SG3 holds.

Since the three subgroup properties hold, $\operatorname{Stab} x$ is a subgroup of G. ■

Exercise E159

Consider again the action of the group $S(\square)$ (see Figure 43) on the set $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9\}$ of modified squares shown below. Write down the stabiliser of each of the modified squares, and check that each of these stabilisers is a subgroup of $S(\square)$.

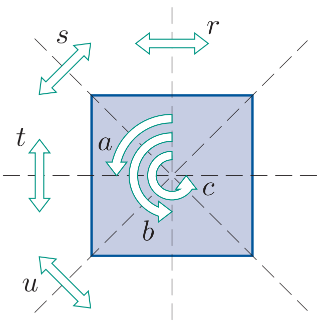
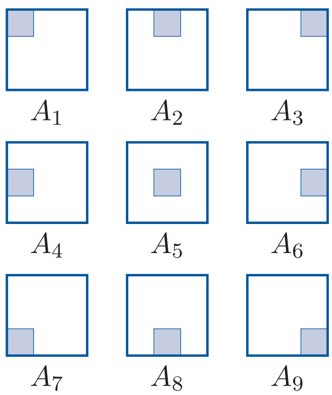
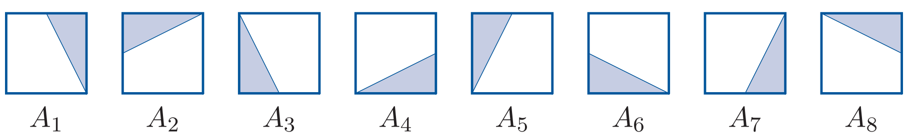


Figure 43 $S(\square)$



Exercise E160

(a) Consider the action of the group $S^+(\square)$ of direct symmetries of the square on the set $X = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ of modified squares shown below. Write down the stabiliser of each of the modified squares under this group action, and check that each of these stabilisers is a subgroup of $S(\square)$.



(b) Now consider the group action of the group $S(\square)$ of all symmetries of the square on the same set X as in part (a). Write down the stabiliser of each of the modified squares under this group action, and check that each of these stabilisers is a subgroup of $S(\square)$.

2.4 Stabilisers of group actions on $\mathbb{R}^2$

In this subsection we will find stabilisers under the group actions on the plane $\mathbb{R}^2$ that we considered in Subsection 2.2.

Worked Exercise E69

Let

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}^+ \right\}.$$

Consider the action $\wedge$ of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \wedge (x, y) = (ax, by)$$

for all $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$. (As in Worked Exercise E64 in Subsection 2.2, you may assume that $(G, \times)$ is a group and that $\wedge$ is a group action.)

Find the stabiliser of each of the following points in $\mathbb{R}^2$.

(a) $(0, 0)$ (b) $(-1, 0)$ (c) $(1, -1)$

Solution

First we find an expression for the stabiliser of a general point (x, y) in $\mathbb{R}^2$ under this group action.

We have to apply the general definition of a stabiliser,

$$\operatorname{Stab} x = \{g \in G : g \wedge x = x\},$$

to the situation here. We

- replace x by a general element of the set $\mathbb{R}^2$, say (x, y)
- replace g by a general element of the group G, say $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

For any point $(x, y) \in \mathbb{R}^2$,

$$\operatorname{Stab}(x, y) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \wedge (x, y) = (x, y) \right\}$$

We use the definition of $\wedge$ to simplify the condition after the colon.

$$= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : (ax, by) = (x, y) \right\}$$

We can rewrite the condition slightly.

$$= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : ax = x \text{ and } by = y \right\}.$$

We now have an expression for the stabiliser of a general point (x, y). We use it to find the stabilisers of the given points.

(a) Putting $(x, y) = (0, 0)$ gives

$$\begin{aligned} \operatorname{Stab}(0, 0) &= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a \times 0 = 0 \text{ and } b \times 0 = 0 \right\} \\ &= G. \end{aligned}$$

(b) Putting $(x, y) = (-1, 0)$ gives

$$\begin{aligned} \operatorname{Stab}(-1, 0) &= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a \times (-1) = -1 \text{ and } b \times 0 = 0 \right\} \\ &= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : -a = -1 \text{ and } 0 = 0 \right\} \\ &= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a = 1 \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} : b \in \mathbb{R}^+ \right\}. \end{aligned}$$

(c) Putting $(x, y) = (1, -1)$ gives

$$\begin{aligned} \operatorname{Stab}(1, -1) &= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a \times 1 = 1 \text{ and } b \times (-1) = -1 \right\} \\ &= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a = 1 \text{ and } b = 1 \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \end{aligned}$$

Notice that the answers to parts (a) and (c) of Worked Exercise E69 are groups under matrix multiplication, as expected in view of Theorem E63. The answer to part (b) is also guaranteed to be a group under matrix multiplication, by the same theorem.

Exercise E161

For the group action in Worked Exercise E69, find the stabiliser of each of the following points.

(a) $(2, 0)$ (b) $(0, 5)$

For a group action on the plane $\mathbb{R}^2$, such as the group action in Worked Exercise E69 and Exercise E161, we cannot of course list the stabiliser of every point in $\mathbb{R}^2$ individually, since there are infinitely many points. However, we can often determine that the stabiliser of each point of a particular form is a certain subgroup of the group that is acting, and the stabiliser of each point of another form is another subgroup of the group, and so on. In this way we may be able to describe the stabiliser of every point in $\mathbb{R}^2$.

In the next exercise you are asked to do this for the group action in Worked Exercise E69 and Exercise E161.

Exercise E162

Consider the group action in Worked Exercise E69 and Exercise E161.

It was found in Worked Exercise E69 that the stabiliser of the origin is the whole group $(G, \times)$.

(a) Show that the stabiliser of every point of the form $(x, 0)$ where $x \in \mathbb{R}^*$ (that is, every point on the x-axis except the origin) is the same subgroup of $(G, \times)$ as found in Exercise E161(a).

(b) Show that the stabiliser of every point of the form $(0, y)$ where $y \in \mathbb{R}^*$ (that is, every point on the y-axis except the origin) is the same subgroup of $(G, \times)$ as found in Exercise E161(b).

(c) Show that the stabiliser of every point of the form (x, y) where $x, y \in \mathbb{R}^*$ (that is, every point that lies neither on the x-axis nor on the y-axis) is the trivial subgroup of $(G, \times)$.

In the next worked exercise we find the stabiliser of every point in $\mathbb{R}^2$ under the group action whose orbits we found in Worked Exercise E65 in Subsection 2.2.

Worked Exercise E70

Consider the action of the group $(\mathbb{R}, +)$ on the set $\mathbb{R}^2$ defined by

$$g \wedge (x, y) = (x + yg, y)$$

for all $g \in \mathbb{R}$ and all $(x, y) \in \mathbb{R}^2$. (You saw that this is a group action in Worked Exercise E59 in Subsection 1.3.)

Find the stabiliser of each point in $\mathbb{R}^2$.

Solution

As in the previous worked exercise, we start by finding an expression for the stabiliser of a general point (x, y) under this group action.

For any point $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
\operatorname{Stab}(x, y) &= \{g \in \mathbb{R} : g \wedge (x, y) = (x, y)\} \\
&= \{g \in \mathbb{R} : (x + yg, y) = (x, y)\} \\
&= \{g \in \mathbb{R} : x + yg = x \text{ and } y = y\} \\
&= \{g \in \mathbb{R} : x + yg = x\} \\
&= \{g \in \mathbb{R} : yg = 0\}.
\end{aligned}$$

We cannot simplify this specification of $\operatorname{Stab}(x, y)$ any further for a general point (x, y). However, for a point (x, y) in which y is non-zero, the condition $yg = 0$ simplifies to $g = 0$, which tells us that the only element of $\operatorname{Stab}(x, y)$ is 0. So we now split into two cases: $y \neq 0$ and $y = 0$.

Hence for any point $(x, y) \in \mathbb{R}^2$ with $y \neq 0$ (that is, any point not on the x-axis),

$$\begin{aligned}
\operatorname{Stab}(x, y) &= \{g \in \mathbb{R} : yg = 0\} \\
&= \{g \in \mathbb{R} : g = 0\} \quad (\text{since } y \neq 0) \\
&= \{0\}.
\end{aligned}$$

Also, for any point $(x, 0) \in \mathbb{R}^2$ (that is, any point on the x-axis),

$$\begin{aligned}
\operatorname{Stab}(x, 0) &= \{g \in \mathbb{R} : 0g = 0\} \\
&= \{g \in \mathbb{R} : 0 = 0\} \\
&= \mathbb{R}.
\end{aligned}$$

We have now found the stabiliser of every point in $\mathbb{R}^2$.

In summary, the stabiliser of any point on the x-axis is the whole group $\mathbb{R}$, and the stabiliser of any other point is the trivial subgroup $\{0\}$.

If you are trying to find the stabiliser of each point in the plane $\mathbb{R}^2$ under a particular group action, and you have found an expression for the stabiliser of a general point (x, y) but are not sure how to proceed from there, then it can be helpful to use your expression to find the stabilisers of a few particular points, just as for orbits. This should help you develop ideas about what happens in general, and you can then try to confirm your ideas algebraically.

Exercise E163

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}.$$

Consider the action of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (ax, y)$$

for all $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$. (You saw that this is a group action in Exercise E143(a) in Subsection 1.4.)

Find the stabiliser of each point in $\mathbb{R}^2$.

Exercise E164

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}.$$

Consider the action of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \wedge (x, y) = (ax, ay)$$

for all $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$. (You saw that this is a group action in Worked Exercise E60 in Subsection 1.4.)

Find the stabiliser of each point in $\mathbb{R}^2$.

3 The Orbit–Stabiliser Theorem

In this section you will meet the *Orbit–Stabiliser Theorem*, an important result that applies to actions of *finite* groups.

3.1 What is the Orbit–Stabiliser Theorem?

We begin with an exercise.

Exercise E165

Consider the action of the group $S(\square)$ (see Figure 44) on the set of all figures in $\mathbb{R}^2$. Complete the following table, in which each row corresponds to the modified square A in $\mathbb{R}^2$ shown at the left of the row. Notice the apparent general relationship between $|\mathrm{Orb}\,A|$ and $|\mathrm{Stab}\,A|$, the numbers of elements in $\mathrm{Orb}\,A$ and $\mathrm{Stab}\,A$, respectively.

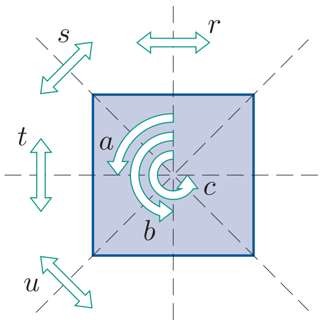


Figure 44 $S(\square)$

A	$\mathrm{Orb}\,A$	$\mathrm{Stab}\,A$	$\lvert\mathrm{Orb}\,A\rvert$	$\lvert\mathrm{Stab}\,A\rvert$
		$\{e, b\}$	4	2
				
				
				

In Exercise E165 you should have found that, for each modified square A in the table,

$$|\mathrm{Orb}\,A| \times |\mathrm{Stab}\,A| = 8.$$

That is, for each of these modified squares, multiplying the number of elements in its orbit by the number of elements in its stabiliser gives the order of the group $S(\square)$. These findings are instances of the following general theorem. It is proved in the next subsection.

Theorem E64 Orbit–Stabiliser Theorem

Suppose that the finite group G acts on the set X. Then, for each element x in X,

$$|\mathrm{Orb}\,x| \times |\mathrm{Stab}\,x| = |G|.$$

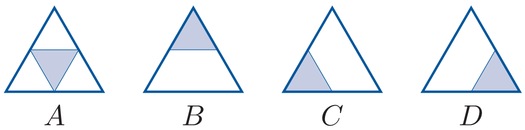
Exercise E166

In each of parts (a), (b) and (c), verify the Orbit–Stabiliser Theorem for each element x of the set on which the group acts.

(a) The action of $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square (see Figure 45).

(b) The action of $S(\square)$ on the set $\{R, S, T, U\}$ of lines of symmetry of the square (see Figure 46).

(c) The action of $S(\triangle)$ (see Figure 47) on the set $\{A, B, C, D\}$ of modified triangles shown below.



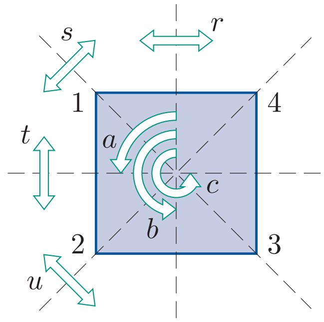


Figure 45 $S(\square)$

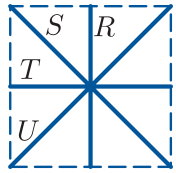


Figure 46 The lines of symmetry of the square

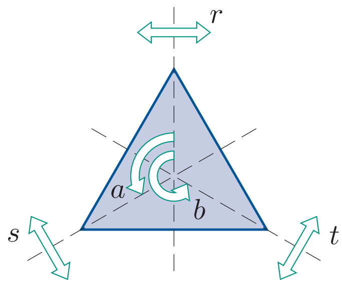


Figure 47 $S(\triangle)$

The Orbit–Stabiliser Theorem has the following immediate corollary.

Corollary E65

Suppose that the finite group G acts on the set X. Then, for each element x in X, the number of elements in $\operatorname{Orb} x$ divides the order of G.

For example, the orbits in the table in the solution to Exercise E165 have 4, 8, 2 and 1 elements, respectively, and these numbers all divide 8, the order of $S(\square)$.

Of course, it also follows from the Orbit–Stabiliser Theorem that if a finite group G acts on a set X, then for each element x in X the number of elements in $\operatorname{Stab} x$ divides the order of G. However, we knew that already: it follows from Lagrange's Theorem, since $\operatorname{Stab} x$ is a subgroup of G.

3.2 Left cosets of stabilisers

Since the stabiliser of a set element under a group action is a subgroup of the group that is acting, it has cosets in this group. In this subsection you will meet an important property of the *left* cosets of stabilisers. We will use this property to prove the Orbit–Stabiliser Theorem.

You might wonder why the property involves left cosets and not right cosets. This is because of the way that we defined a group action. The concept that we have been calling a group action is more accurately called a *left group action*.

In the definition of a group action that you met earlier, axiom GA3 is as follows:

GA3 Composition For all $g, h \in G$ and all $x \in X$,

$$g \wedge (h \wedge x) = (g \circ h) \wedge x.$$

If we replace axiom GA3 with the following alternative axiom, then we obtain the definition of a *right group action*:

GA3 Composition (different) For all $g, h \in G$ and all $x \in X$,

$$g \wedge (h \wedge x) = (h \circ g) \wedge x.$$

If we had used this alternative definition, then we would have obtained a theory analogous to the one developed in this unit, just with a few things 'the other way round'. The situation is similar to that for left and right cosets. We will continue to use our original axiom GA3 throughout this unit – that is, we will continue to work with left group actions, and call them simply group actions.

Here is an example that illustrates the property of left cosets of stabilisers introduced in this subsection. Consider the action of the group $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square (see Figure 48), and consider in particular the vertex label 1.

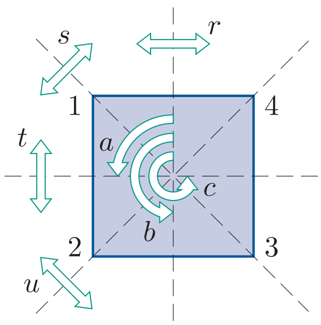


Figure 48 $S(\square)$

Table 1 $S(\square)$

$\circ$	e	a	b	c	r	s	t	u
e	e	a	b	c	r	s	t	u
a	a	b	c	e	s	t	u	r
b	b	c	e	a	t	u	r	s
c	c	e	a	b	u	r	s	t
r	r	u	t	s	e	c	b	a
s	s	r	u	t	a	e	c	b
t	t	s	r	u	b	a	e	c
u	u	t	s	r	c	b	a	e

The elements of $S(\square)$ that fix 1 are e and s, so

$$\operatorname{Stab} 1 = \{e, s\}.$$

We will now find the left cosets of Stab 1 in $S(\square)$. Using our usual method for finding cosets and referring to Table 1, we find that they are

$$\begin{aligned}
\operatorname{Stab} 1 &= \{e, s\},\\
a \operatorname{Stab} 1 &= \{a \circ e, a \circ s\} = \{a, t\},\\
b \operatorname{Stab} 1 &= \{b \circ e, b \circ s\} = \{b, u\},\\
c \operatorname{Stab} 1 &= \{c \circ e, c \circ s\} = \{c, r\}.
\end{aligned}$$

Now let us partition $S(\square)$ in another way, namely according to where its elements map the vertex label 1. We can see from Figure 48 that

e and s map 1 to 1 (of course, since $e, s \in \operatorname{Stab} 1$),
a and t map 1 to 2,
b and u map 1 to 3,
c and r map 1 to 4.

So the partition of $S(\square)$ according to where its elements map 1 is

$$\{e, s\}, \quad \{a, t\}, \quad \{b, u\}, \quad \{c, r\}.$$

This is the same as the partition of $S(\square)$ into left cosets of Stab 1.

So if two elements of $S(\square)$ lie in the *same* left coset of Stab 1, then they map 1 to the *same* vertex label, whereas if they lie in *different* left cosets, then they map 1 to *different* vertex labels.

In the next exercise you are asked to determine whether a similar property holds for the vertex label 2 under the same group action.

Exercise E167

Consider the action of the group $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square (see Figure 49).

(a) Find $\operatorname{Stab} 2$.

(b) Find the left cosets of $\operatorname{Stab} 2$ in $S(\square)$. (The group table of $S(\square)$ is given as Table 2.)

(c) Partition $S(\square)$ according to where its elements map the vertex label 2.

(d) Are the partitions that you found in parts (b) and (c) the same?

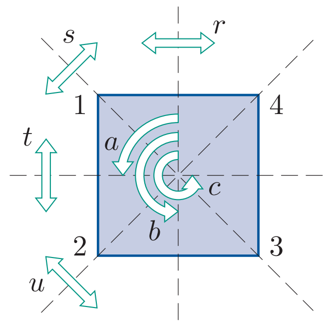


Figure 49 $S(\square)$

Table 2 $S(\square)$

$\circ$	e	a	b	c	r	s	t	u
e	e	a	b	c	r	s	t	u
a	a	b	c	e	s	t	u	r
b	b	c	e	a	t	u	r	s
c	c	e	a	b	u	r	s	t
r	r	u	t	s	e	c	b	a
s	s	r	u	t	a	e	c	b
t	t	s	r	u	b	a	e	c
u	u	t	s	r	c	b	a	e

The examples above are instances of the following general result, which is illustrated in Figure 50.

Theorem E66

Let $\wedge$ be an action of a group $(G, \circ)$ on a set X, let x be an element of X and let g and h be elements of G. Then

$$g \wedge x = h \wedge x$$

if and only if

g and h lie in the same left coset of $\operatorname{Stab} x$.

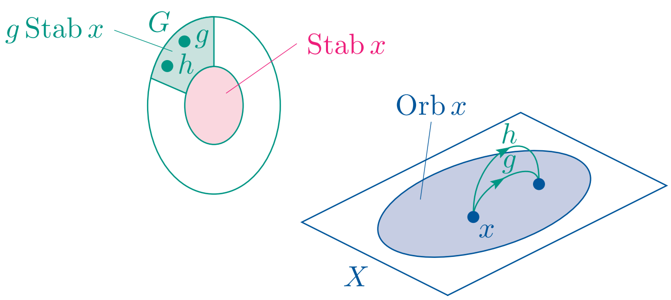


Figure 50 Group elements g and h map set element x to the same element if and only if they lie in the same left coset of $\operatorname{Stab} x$

Proof of Theorem E66 **'If' part**

Suppose that g and h lie in the same left coset of $\operatorname{Stab} x$. Then $h \in g \operatorname{Stab} x$, so $h = g \circ k$ for some $k \in \operatorname{Stab} x$. It follows that

$$\begin{aligned} h \wedge x &= (g \circ k) \wedge x \\ &= g \wedge (k \wedge x) \quad \text{(by axiom GA3)} \\ &= g \wedge x \quad (\text{since } k \in \operatorname{Stab} x), \end{aligned}$$

as required.

'Only if' part

Suppose that

$$g \wedge x = h \wedge x.$$

Consider the effect of the group element $g^{-1} \circ h$ on x:

$$\begin{aligned}(g^{-1} \circ h) \wedge x &= g^{-1} \wedge (h \wedge x) \quad \text{(by axiom GA3)}\\ &= g^{-1} \wedge (g \wedge x) \quad (\text{since } g \wedge x = h \wedge x)\\ &= (g^{-1} \circ g) \wedge x \quad \text{(by axiom GA3)}\\ &= e \wedge x\\ &= x \quad \text{(by axiom GA2).}\end{aligned}$$

Therefore $g^{-1} \circ h = k$ for some $k \in \operatorname{Stab} x$. It follows, by composing each side of this equation on the left by g, that $h = g \circ k$. Hence $h \in g \operatorname{Stab} x$. Thus g and h lie in the same left coset of $\operatorname{Stab} x$. ■

Theorem E66 tells us that if a group G acts on a set X and x is any element of X, then the sets of group elements that map x to the same element of X are precisely the left cosets of $\operatorname{Stab} x$. This means that

> if we collect together the group elements according to where they map x, then we have the left cosets of $\operatorname{Stab} x$,

and that, conversely,

> if we find the left cosets of $\operatorname{Stab} x$, then we have the sets of group elements that map x to the same element of X.

In the next exercise you are asked to check Theorem E66 for a set element under another group action.

Exercise E168

Consider the action of the symmetric group S_3 on the set $\{1, 2, 3\}$ of symbols.

(a) Find Stab 1.

(b) Find the left cosets of Stab 1.

(c) Partition S_3 according to where its elements map the symbol 1.

(d) Check whether the partitions that you found in parts (b) and (c) are the same.

Although the examples illustrating Theorem E66 that you have seen so far all involve actions of finite groups on finite sets, the theorem applies to *all* group actions, no matter whether the group and set involved are finite or infinite.

We can now use Theorem E66 to prove the Orbit–Stabiliser Theorem. The proof is based on the following idea. Consider any action of a group G on a set X, and let x be an element of X. By Theorem E66, the sets of elements of G that map x to the same element are precisely the left cosets of $\operatorname{Stab} x$ in G. It follows that we can define a mapping, say f, whose domain is the set of left cosets of $\operatorname{Stab} x$ in G, whose codomain is $\operatorname{Orb} x$, and whose rule is

left coset $\longmapsto$ element of $\operatorname{Orb} x$ to which each element of the left coset maps x.

This mapping f is illustrated in Figure 51.

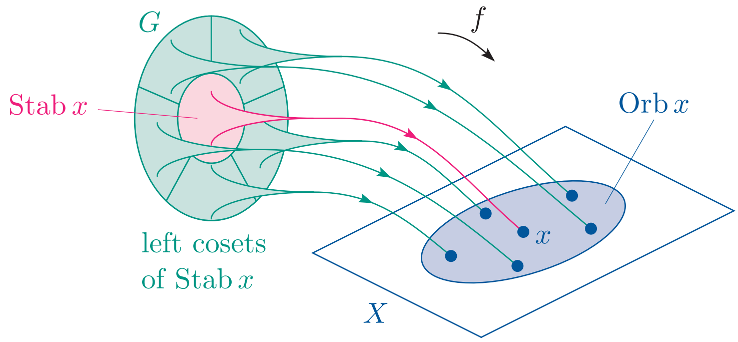


Figure 51 The mapping f obtained from the stabiliser of an element x

For example, consider again the action of the group $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square, and the particular vertex label 1.

Near the start of this subsection you saw that under this group action the left cosets of $\operatorname{Stab} 1$ are

$$\{e, s\}, \quad \{a, t\}, \quad \{b, u\}, \quad \{c, r\}.$$

You also saw that

both elements of the left coset $\{e, s\}$ ($\operatorname{Stab} 1$ itself) map 1 to 1,
both elements of the left coset $\{a, t\}$ map 1 to 2,
both elements of the left coset $\{b, u\}$ map 1 to 3,
both elements of the left coset $\{c, r\}$ map 1 to 4.

So the mapping f obtained from $\operatorname{Stab} 1$ as described above is

$$\begin{aligned} f : \text{set of left cosets of } \operatorname{Stab} 1 &\longrightarrow \operatorname{Orb} 1 \\ \{e, s\} &\longmapsto 1 \\ \{a, t\} &\longmapsto 2 \\ \{b, u\} &\longmapsto 3 \\ \{c, r\} &\longmapsto 4 \end{aligned}$$

This mapping f is illustrated in Figure 52.

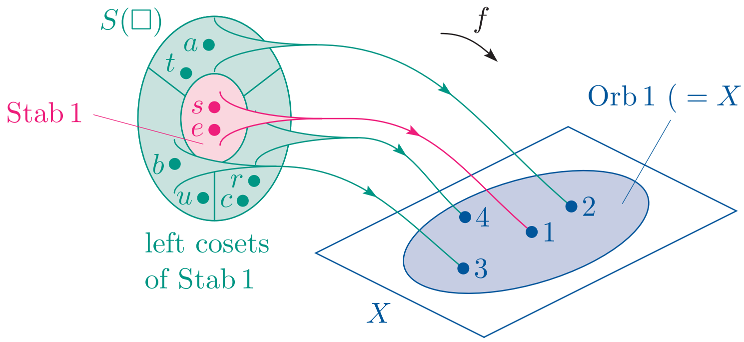


Figure 52 The mapping f obtained from Stab 1 under the action of $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square

In this example the orbit of the set element considered, Orb 1, is the whole of the set X on which the group acts, but in other examples the orbit may be a proper subset of X.

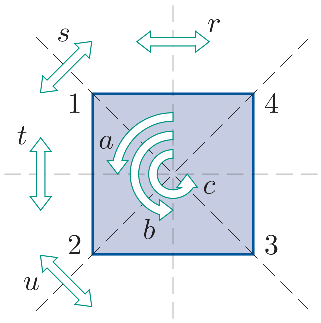


Figure 53 $S(\square)$

Exercise E169

Consider again the action of the group $S(\square)$ on the set of vertex labels of the square (see Figure 53). By referring to your solution to Exercise E167, write down the mapping f obtained from Stab 2 in the way described above.

The mapping f obtained from the stabiliser of a set element under a group action in the way described above is always one-to-one and onto, just because of how it is defined. This fact is stated formally in the following corollary to Theorem E66. It is the key to proving the Orbit–Stabiliser Theorem, as you will see shortly.

Corollary E67

Let $\wedge$ be an action of a group G on a set X and let x be an element of X. Then the mapping f given by

$$\begin{aligned} f : \text{set of left cosets of } \operatorname{Stab} x &\longrightarrow \operatorname{Orb} x \\ g \operatorname{Stab} x &\longmapsto g \wedge x \end{aligned}$$

is one-to-one and onto.

Proof The mapping f defined above maps each left coset of $\operatorname{Stab} x$ to $g \wedge x$, where g is any element of the left coset. This is a valid definition of a mapping because, by Theorem E66, $g \wedge x$ is the *same* element of X for *every* group element g in any particular left coset of $\operatorname{Stab} x$.

Theorem E66 tells us that elements from *different* left cosets of $\operatorname{Stab} x$ map x to *different* elements of X, so f is one-to-one.

Also, f is onto, because each element $g \wedge x$ of $\operatorname{Orb} x$ is the image under f of the left coset $g \operatorname{Stab} x$. ■

We now use Corollary E67 to prove the Orbit–Stabiliser Theorem. Unlike Theorem E66 and Corollary E67, the Orbit–Stabiliser Theorem is a result about *finite* groups only.

Theorem E64 Orbit–Stabiliser Theorem

Suppose that the finite group G acts on the set X. Then, for each element x in X,

$$|\operatorname{Orb} x| \times |\operatorname{Stab} x| = |G|.$$

Proof Let x be an element of X. Corollary E67 tells us that the left cosets of $\operatorname{Stab} x$ can be matched one-to-one with the elements of $\operatorname{Orb} x$. Hence the number of left cosets of $\operatorname{Stab} x$ is the same as the number of elements in $\operatorname{Orb} x$. But the number of left cosets of $\operatorname{Stab} x$ is equal to $|G|/|\operatorname{Stab} x|$, so

$$|G|/|\operatorname{Stab} x| = |\operatorname{Orb} x|,$$

and hence

$$|\operatorname{Orb} x| \times |\operatorname{Stab} x| = |G|.$$ ■

3.3 Groups acting on groups

In this subsection we will look at some examples of actions of a group G on a set X where X is itself a group. Often, but not always, the set X is the group G itself. Such actions have important applications in group theory, as you will see.

Throughout the subsection we will mostly use concise multiplicative notation for abstract groups: that is, we will not use symbols for their binary operations. This is convenient when we have to deal with many composites of group elements, as you have seen before.

The definition of a group action is translated into concise multiplicative notation below. There are only two differences: we refer to the group as G instead of $(G, \circ)$, and in axiom GA3 we write $(gh) \wedge x$ instead of $(g \circ h) \wedge x$.

Definition

Let G be a group with identity element e, and let X be a set. Suppose that for each element g in G and each element x in X an object $g \wedge x$ is defined in some way.

We say that the effect $\wedge$ of G on X is a **group action** of G on X, or simply an **action** of G on X, and that G **acts on** X, if the following three axioms hold.

GA1 Closure For each $g \in G$ and each $x \in X$,

$$g \wedge x \in X.$$

GA2 Identity For each $x \in X$,

$$e \wedge x = x.$$

GA3 Composition For all $g, h \in G$ and all $x \in X$,

$$g \wedge (h \wedge x) = (gh) \wedge x.$$

The first action of a group on a group that we consider in this subsection is *conjugation*. The proposition below shows that conjugation is an action of a group on itself.

Proposition E68

Let G be a group, and let $\wedge$ be defined by

$$g \wedge x = gxg^{-1}$$

for all $g, x \in G$. Then $\wedge$ is an action of G on itself.

Proof We show that the group action axioms hold.

GA1 Closure Let $g, x \in G$. Then

$$g \wedge x = gxg^{-1} \in G.$$

Thus axiom GA1 holds.

GA2 Identity Let e be the identity element of G and let $x \in G$. Then

$$e \wedge x = exe^{-1} = x.$$

Thus axiom GA2 holds.

GA3 Composition Let $g, h, x \in G$. We have to check that

$$g \wedge (h \wedge x) = (gh) \wedge x.$$

Now

$$\begin{aligned} g \wedge (h \wedge x) &= g \wedge (hxh^{-1}) \\ &= ghxh^{-1}g^{-1} \\ &= (gh)x(gh)^{-1} \quad (\text{since } h^{-1}g^{-1} = (gh)^{-1}) \\ &= (gh) \wedge x. \end{aligned}$$

Thus axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action. ■

Exercise E170

Let G be a group. Determine which of the following define a group action $\wedge$ of G on itself.

(a) $g \wedge x = gx$ for all $g, x \in G$.

(b) $g \wedge x = xg$ for all $g, x \in G$.

(c) $g \wedge x = xg^{-1}$ for all $g, x \in G$.

We will now revisit some topics in group theory in the light of group actions.

Lagrange's Theorem

Lagrange's Theorem is related to the group action defined in the exercise below, as you will see. This group action is slightly different from those that you have met so far in this section: it does not necessarily involve an action of a group on itself, but instead involves an action of a *subgroup* on the group.

Exercise E171

Let H be a subgroup of a group G. Let $\wedge$ be defined by

$$h \wedge g = hg$$

for all $h \in H$ and all $g \in G$. Show that $\wedge$ defines an action of H on G.

Now let H be a subgroup of a group G, and consider the action of H on G defined in Exercise E171. Let us investigate its orbits. For any element g of G,

$$\operatorname{Orb} g = \{h \wedge g : h \in H\} = \{hg : h \in H\}.$$

This is just the right coset Hg. So the orbits of this group action are precisely the *right cosets of H in G*. Hence the partition of a group G into the right cosets of a subgroup H is a particular instance of the partition of a group into the orbits of a group action.

Now suppose that the group G is finite. Corollary E65, an immediate corollary of the Orbit–Stabiliser Theorem, states that the number of elements in an orbit of an action of a finite group divides the order of the group. Applying this result to the group action above tells us that the number of elements in each right coset of H in G divides the order of G. Since H is one of the right cosets, this tells us that the order of H divides the order of G. This is Lagrange's Theorem, so Lagrange's Theorem is a special case of Corollary E65.

Conjugacy classes

We will now use group actions to prove a theorem about *conjugacy classes* that was stated but not proved in Unit E2 *Quotient groups and conjugacy*.

You saw in Unit E2 that every group splits into conjugacy classes: elements in the same conjugacy class are conjugate to each other in the group, and elements in different classes are not conjugate to each other in the group.

For example, the conjugacy classes of the symmetry group $S(\triangle)$ (see Figure 54) are as follows (they were found in Exercise E77 in Subsection 2.3 of Unit E2):

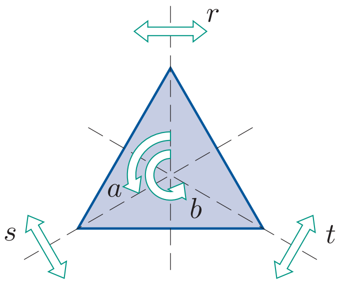


Figure 54 $S(\triangle)$

$\{e\}$	identity
$\{a, b\}$	anticlockwise and clockwise rotations through $2\pi/3$
$\{r, s, t\}$	reflections in lines through vertices and midpoints of edges.

You met the following theorem in Subsection 2.3 of Unit E2.

Theorem E27

In any finite group G, the number of elements in each conjugacy class divides the order of G.

For instance, the numbers of elements in the conjugacy classes of $S(\triangle)$ are 1, 2 and 3, respectively, and each of these numbers divides 6, the order of $S(\triangle)$.

We can now use the group action defined in Proposition E68 earlier in this subsection to prove Theorem E27.

Proof of Theorem E27 Let G be a finite group, and let $\wedge$ be defined by

$$g \wedge x = gxg^{-1}$$

for all $g, x \in G$. By Proposition E68, $\wedge$ is an action of G on itself.

For any element x in G, the orbit of x under $\wedge$ is

$$\begin{aligned}\operatorname{Orb} x &= \{g \wedge x : g \in G\}\\ &= \{gxg^{-1} : g \in G\}.\end{aligned}$$

This is the conjugacy class of G containing x.

Thus the orbits of this group action are precisely the conjugacy classes of G. By Corollary E65 to the Orbit–Stabiliser Theorem, the number of elements in each orbit divides the order of G, which proves the result. ■

Homomorphisms

We can also recognise a result about homomorphisms from Unit E3 as a special case of the Orbit–Stabiliser Theorem. To do this, we apply the Orbit–Stabiliser Theorem to the group action in the next exercise.

Exercise E172

Let $\phi : (G, \circ) \longrightarrow (H, *)$ be a homomorphism. Let $\wedge$ be defined by

$$g \wedge h = \phi(g) * h,$$

for all $g \in G$ and $h \in H$. (Notice that it is the binary operation of the group $(H, *)$ that is used in the definition of $\wedge$.)

Show that $\wedge$ is an action of the group $(G, \circ)$ on the group $(H, *)$.

Now let $\phi : (G, \circ) \longrightarrow (H, *)$ be a homomorphism where $(G, \circ)$ is a *finite* group, and let $\wedge$ be the action of $(G, \circ)$ on $(H, *)$ defined in Exercise E172.

Let us find the orbit and stabiliser of e_H, the identity element of $(H, *)$, under this group action.

The orbit of e_H is

$$\begin{aligned}\operatorname{Orb} e_H &= \{g \wedge e_H : g \in G\} \\ &= \{\phi(g) * e_H : g \in G\} \\ &= \{\phi(g) : g \in G\}.\end{aligned}$$

This set is the *image* of ϕ. So $\operatorname{Orb} e_H = \operatorname{Im} \phi$.

The stabiliser of e_H is

$$\begin{aligned}\operatorname{Stab} e_H &= \{g \in G : g \wedge e_H = e_H\} \\ &= \{g \in G : \phi(g) * e_H = e_H\} \\ &= \{g \in G : \phi(g) = e_H\}.\end{aligned}$$

This set is the *kernel* of ϕ. So $\operatorname{Stab} e_H = \operatorname{Ker} \phi$.

By the Orbit–Stabiliser Theorem,

$$|\operatorname{Orb} e_H| \times |\operatorname{Stab} e_H| = |G|.$$

Therefore

$$|\operatorname{Im} \phi| \times |\operatorname{Ker} \phi| = |G|.$$

This is Corollary E56 from Unit E3 – it is a corollary of the First Isomorphism Theorem. Thus Corollary E56 is a special case of the Orbit–Stabiliser Theorem.

Group actions can be used to prove many other results in group theory. The examples that you have seen in this subsection illustrate the power of this approach.

4 The Counting Theorem

A **counting problem** is a problem that asks how many objects there are of a particular type. In this section you will learn how to solve some counting problems that involve symmetry. Many problems of this kind look hard to answer at first sight, but become much more straightforward if we apply ideas relating to group actions.

4.1 Counting problems involving symmetry

Some simple counting problems are easily solved by using the following rule.

Multiplication Principle

If we have k successive choices to make, and the ith choice involves choosing from n_i options, for each $i = 1, 2, \ldots, k$, then the total number of ways to make all k choices is

$$n_1 \times n_2 \times \cdots \times n_k.$$

Here is an example.

Worked Exercise E71

How many distinct sequences of two coloured discs are there in which each disc is coloured blue, yellow or red? Some examples of such sequences are shown below.

Solution

There are three choices of colour for the first disc and three choices of colour for the second disc, so by the Multiplication Principle the number of such sequences is

$$3 \times 3 = 9.$$

The nine sequences from Worked Exercise E71 are shown in Figure 55.

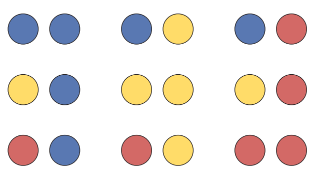
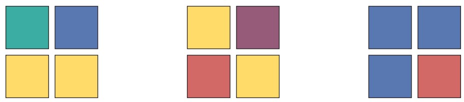


Figure 55 The different sequences of two discs coloured blue, yellow or red

Exercise E173

A 2×2 pattern of plain coloured tiles is to be mounted on a wall. How many different patterns are possible if tiles are available in blue, yellow, red, green and purple? Some examples of such patterns are shown below.



Now consider the following counting problem.

> **Bangle problem** How many different bangles decorated with six equally spaced beads can be made if beads are available in blue, yellow and red? Some examples of such bangles are shown in Figure 56.

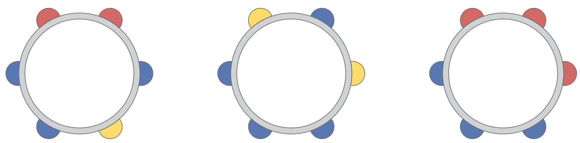


Figure 56 Six-bead bangles made using blue, yellow and red beads

If the bangles in this problem cannot be rotated or turned over – that is, if their positions are fixed – then we can answer this question by using the Multiplication Principle. There are six beads, and each of them can be any of the three colours, so the number of different bangles is

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729.$$

However, such a bangle *can* be rotated or turned over, of course. For example, we would regard the two bangles in Figure 57 as the same, since either can be rotated to give the other.

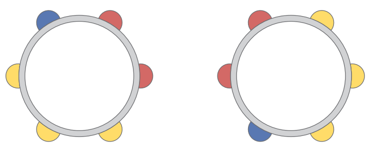


Figure 57 Two bangles each of which can be rotated to give the other

Similarly, we would regard the two bangles in Figure 58 as the same, since either can be turned over to give the other.

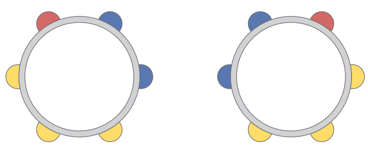
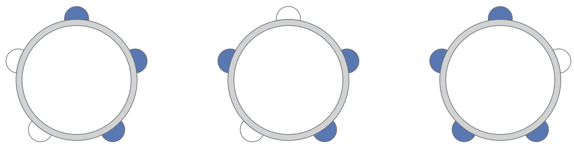


Figure 58 Two bangles each of which can be turned over to give the other

The symmetry of the objects involved in this counting problem makes it much more difficult to solve than the problems in Worked Exercise E71 and Exercise E173.

Exercise E174

Consider the problem of finding the number of different bangles that can be made using *five* equally spaced beads, if beads are available in just *two* colours, blue and white. Some examples of such bangles are shown below.



(a) How many different bangles are there if the bangles cannot be rotated or turned over?

(b) By drawing all the possibilities, find the number of different bangles if two bangles are regarded as the same whenever one can be rotated or turned over to give the other.

What has all this got to do with group actions? To see this, consider again the original bangle problem above, which concerned six-bead bangles made using beads available in three colours. Let X be the set of all 3^6 coloured bangles in fixed positions. We can think of the beads on each bangle as being placed at the vertices of a regular hexagon, and we can think of turning a bangle over as reflecting it, so the symmetry group of the bangle (when we ignore the colours of the beads) is essentially the symmetry group $S(\hexagon)$ of the regular hexagon. The rotations and reflections in $S(\hexagon)$ map bangles in X to other bangles in X, and the effect of $S(\hexagon)$ on X is a group action by Theorem E60.

We want to regard two bangles in the set X as the same if either can be rotated or reflected to give the other. In other words, we want to regard two bangles as the same if they *lie in the same orbit* of the action of the group $S(\hexagon)$ on the set X. Thus the bangle problem can be rephrased as follows.

> Let X be the set of all possible bangles in fixed positions decorated with six equally spaced beads each coloured blue, yellow or red. How many orbits are there in the action of the group $S(\hexagon)$ on the set X?

Later in this section you will meet a theorem, the *Counting Theorem*, that gives a formula for the number of orbits of an action of a finite group on a finite set. You will see how to use it to answer counting problems such as the one above.

First, however, we will look at a few more counting problems that illustrate the kinds of questions that we can answer by using the Counting Theorem. Here is another example of such a problem.

> **Chessboard problem** How many different patterns can be made by colouring the squares of a chessboard either black or white?

If the chessboard in this problem is fixed in place – for example, if it is displayed on a wall – then we can answer this question by using the

Multiplication Principle, as follows. A chessboard has 64 squares and each square can be coloured with either of two colours, so the total number of coloured chessboards is

$$\underbrace{2 \times 2 \times 2 \times \cdots \times 2}_{\text{64 copies of 2}} = 2^{64}.$$

However, usually a chessboard can be rotated, so we would want to regard two coloured chessboards as the same if one can be rotated to give the other. For example, we would want to regard the two coloured chessboards in Figure 59 as the same, as a quarter turn anticlockwise turns the first into the second. A chessboard usually appears on only one side of its board, so we would *not* want to regard two coloured chessboards as the same if one can be reflected to give the other (except in cases where one can also be rotated to give the other, of course).

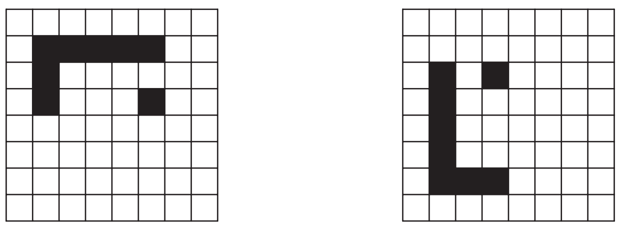


Figure 59 Two coloured chessboards each of which can be rotated to give the other

In the next exercise you are asked to look at a similar but smaller problem.

Exercise E175

There are $2^4 = 16$ ways of colouring the squares of a 2×2 chessboard in a fixed position either black or white (by the Multiplication Principle). Three of them are shown below.

(a) Draw all 16 coloured chessboards in fixed positions.

(b) By using your drawings, determine how many different such coloured chessboards there are when we regard two of them as the same if one can be rotated to give the other.

Like the bangle problem, the chessboard problem can be interpreted in terms of a group action. Let X be the set of all 2^{64} coloured chessboards in fixed positions. We want to regard two coloured chessboards as the same when one can be rotated to give the other, so we consider the action of the group $S^+(\square)$ of rotations of the square on the set X. Then we are regarding two coloured chessboards as the same when they belong to the same orbit of this group action, so the answer to the chessboard problem is the number of orbits of the group action.

Finally consider the following counting problem.

Cube problem How many different coloured cubes are there with each face painted one of blue, yellow or red?

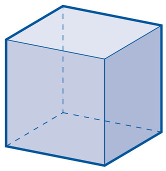


Figure 60 A cube

A cube (see Figure 60) has six faces, and in this problem each of them is to be coloured with one of three colours, so by the Multiplication Principle there are 3^6 coloured cubes in fixed positions. However, we would want to regard two coloured cubes as the same when one can be rotated to give the other.

We can interpret this problem in terms of a group action as follows. We let X be the set of all 3^6 coloured cubes in fixed positions. We want to regard two coloured cubes as the same when one can be rotated to give the other, so we consider the action of the group $S^+(\text{cube})$ of rotations of the cube on the set X. Then we are regarding two coloured cubes as the same when they belong to the same orbit of this group action, so the answer to the cube problem is the number of orbits of the group action.

There is one more concept relating to group actions that you need to learn about before you can meet the Counting Theorem and discover how to solve problems such as those in this subsection. This is the concept of *fixed sets*, which is covered in the next subsection.

4.2 Fixed sets

We make the following definition.

Definition

Let $\wedge$ be an action of a group G on a set X, and let g be an element of G. The **fixed set** of g under $\wedge$, denoted by $\operatorname{Fix} g$, is given by

$$\operatorname{Fix} g = \{x \in X : g \wedge x = x\}.$$

That is, $\operatorname{Fix} g$ is the set of elements of X that are fixed by g.

This definition is illustrated in Figure 61.

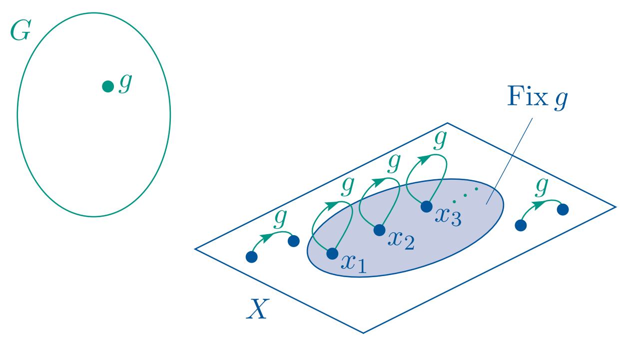


Figure 61 The fixed set of a group element g

Notice that it is an element of the *group* G, not an element of the *set* X, that has a fixed set. This is in contrast to orbits and stabilisers, each of which is a set associated with an element of the set X. Fixed sets, like stabilisers, are concerned with elements of the group G fixing elements of the set X, but from the opposite point of view:

- the fixed set of an element g in G is the set of all elements of X that are fixed by g
- the stabiliser of an element x in X is the set of all elements of G that fix x.

In particular, $\operatorname{Fix} g$ is a subset of X, whereas $\operatorname{Stab} x$ is a subgroup of G.

The fixed set of the identity element e of the group G is always the whole set X, since by axiom GA2 the identity element e fixes every element of X.

Worked Exercise E72

Consider the action of the group $S(\square)$ (see Figure 62) on the set $\{R, S, T, U\}$ of lines of symmetry of the square (see Figure 63). Write down the fixed set of each element of $S(\square)$ under this group action.

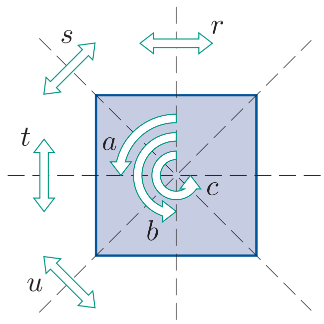


Figure 62 $S(\square)$

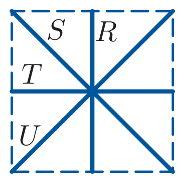


Figure 63 The lines of symmetry of the square

Solution

To find $\operatorname{Fix} r$, for example, we consider the effect of the transformation r on each element of $\{R, S, T, U\}$:

$$r \wedge R = R,$$
$$r \wedge S = U,$$
$$r \wedge T = T,$$
$$r \wedge U = S.$$

The elements of $\{R, S, T, U\}$ that are fixed by r are R and T, so $\operatorname{Fix} r = \{R, T\}$. We find the fixed sets of the other elements of $S(\square)$ in a similar way.

The fixed sets are

$$\operatorname{Fix} e = \{R, S, T, U\},$$
$$\operatorname{Fix} a = \varnothing \quad \text{(the empty set)},$$
$$\operatorname{Fix} b = \{R, S, T, U\},$$
$$\operatorname{Fix} c = \varnothing,$$
$$\operatorname{Fix} r = \{R, T\},$$
$$\operatorname{Fix} s = \{S, U\},$$
$$\operatorname{Fix} t = \{R, T\},$$
$$\operatorname{Fix} u = \{S, U\}.$$

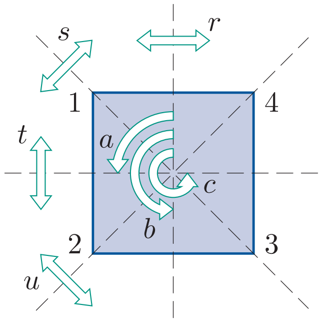


Figure 64 $S(\square)$

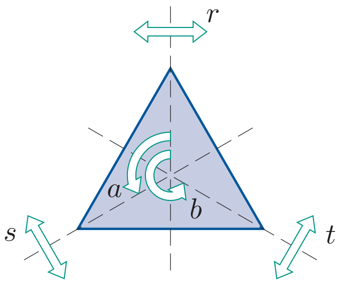


Figure 65 $S(\triangle)$

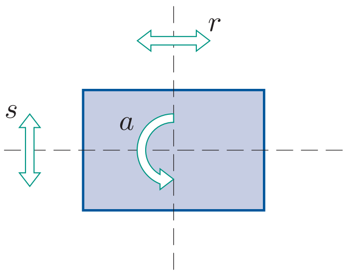


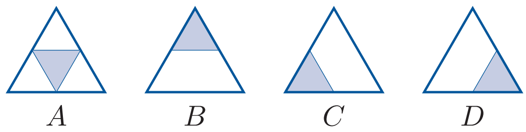
Figure 66 $S(\sqsubset\!\sqsupset)$

Exercise E176

Write down the fixed set of each element of the group $S(\square)$ under the action of $S(\square)$ on the set $\{1, 2, 3, 4\}$ of vertex labels of the square (see Figure 64).

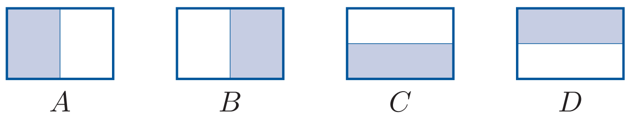
Exercise E177

Write down the fixed set of each element of the group $S(\triangle)$ (see Figure 65) under the action of $S(\triangle)$ on the set $\{A, B, C, D\}$ of modified triangles shown below.



Exercise E178

Write down the fixed set of each element of the group $S(\sqsubset\!\sqsupset)$ (see Figure 66) under the action of $S(\sqsubset\!\sqsupset)$ on the set $\{A, B, C, D\}$ of modified rectangles shown below.



The fixed point sets that you met in Subsection 4.1 of Unit E2 are special cases of fixed sets. You saw there that if f is a symmetry of a figure F, then the *fixed point set* of f is the set of all points of F that are fixed by f. This is the fixed set of f under the natural action of the symmetry group $S(F)$ on the set of points in F.

The next worked exercise and the exercise that follows involve finding fixed sets under the action of a group on the plane $\mathbb{R}^2$.

Worked Exercise E73

Let

$$G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}^+ \right\}.$$

Consider the action of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \wedge (x, y) = (ax, by)$$

for all $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$.

(This is the same group action as in Worked Exercise E64 in Subsection 2.2 and Worked Exercise E69 in Subsection 2.4.)

(a) Find an expression for the fixed set of a general element of the group $(G, \times)$ under this group action.

(b) Find the fixed set of each of the following elements of $(G, \times)$ under the group action. Describe each fixed set geometrically.

(i) $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

(a) We have to apply the general definition of a fixed set,

$$\operatorname{Fix} g = \{x \in X : g \wedge x = x\},$$

to the situation here. We

- replace g by a general element of the group G, say $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
- replace x by a general element of the set $\mathbb{R}^2$, say (x, y).

For any matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G$ (so $a, b \in \mathbb{R}^+$),

$$\begin{aligned} \operatorname{Fix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} &= \left\{(x, y) \in \mathbb{R}^2 : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \wedge (x, y) = (x, y)\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : (ax, by) = (x, y)\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : ax = x \text{ and } by = y\right\}. \end{aligned}$$

(b) (i) By part (a),

$$\begin{aligned} \operatorname{Fix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} &= \left\{(x, y) \in \mathbb{R}^2 : 1x = x \text{ and } 3y = y\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\} \\ &= \{(x, 0) : x \in \mathbb{R}\}. \end{aligned}$$

So this fixed set is the x-axis.

(ii) Here the given matrix is the identity element of $(G, \times)$. Under any group action the identity element of the group fixes every element of the set, by axiom GA2. So there is no need to use the formula from part (a) here (though of course it would give the same answer).

Since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element of G,

$$\operatorname{Fix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{R}^2.$$

That is, this fixed set is the whole plane $\mathbb{R}^2$.

Exercise E179

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}.$$

Consider the action of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (ax, y)$$

for all $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$. (You saw that this is a group action in Exercise E143(a) in Subsection 1.4.)

(a) Find an expression for the fixed set of a general element of the group $(G, \times)$ under this group action.

(b) Find the fixed set of each of the following elements of $(G, \times)$ under the group action. Describe each fixed set geometrically.

(i) $\begin{pmatrix} -1 & 5 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

You will need to use the idea of fixed sets in the next subsection, where you will meet the Counting Theorem and see how to use it to solve counting problems involving symmetry. Using the Counting Theorem in this way usually involves considering the action of a finite group G of symmetries on a large finite set X of coloured figures. To be able to apply the Counting Theorem we need to know the *sizes* of the fixed sets of the elements of G, that is, the numbers of elements that the fixed sets contain. So we will now look at how we can find the sizes of fixed sets in this sort of situation. Here is an example.

Worked Exercise E74

Consider the action of the group $S(\triangle)$ (see Figure 67) on the set X whose elements are all the coloured figures obtained by colouring each of the four small triangles in the figure on the left below with one of the three colours blue, yellow and red. Some examples of elements of the set X are shown on the right below.

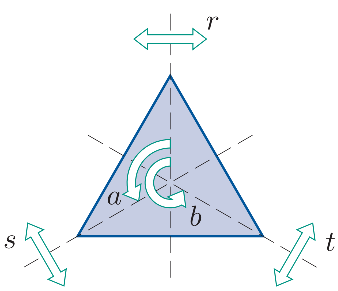
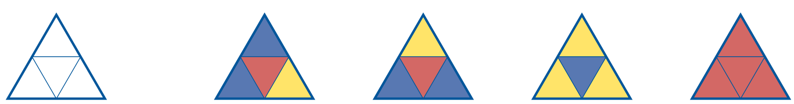


Figure 67 $S(\triangle)$



Find the size of $\text{Fix}\, g$ for each symmetry g in $S(\triangle)$.

Solution

We consider each symmetry in $S(\triangle)$ in turn.

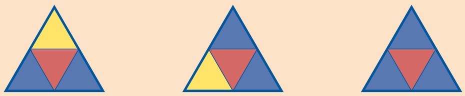
- First consider the identity symmetry e. It fixes all the coloured figures in X.

  There are four small triangles, each coloured with one of three colours, so the number of coloured figures in X is 3^4 (by the Multiplication Principle). Hence

  $$|\text{Fix}\, e| = 3^4.$$

- Now consider the symmetry a.

  Let us think about the effect of a on some coloured figures in X. The symmetry a does not fix the first coloured figure below, because it maps it to the second coloured figure, which is different. However, it does fix the third coloured figure.

  In general, we can say the following.

  The coloured figures in X fixed by the symmetry a are those in which the three outer triangles are all the same colour.

  For such a coloured figure, there are three choices for the colour of the middle triangle and three choices for the single colour of the three outer triangles, so the number of such coloured figures is 3^2. Hence

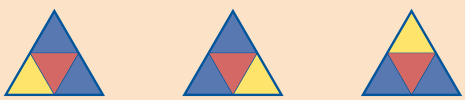
  $$|\text{Fix}\, a| = 3^2.$$

  By a similar argument,

  $$|\text{Fix}\, b| = 3^2.$$

- Now consider the symmetry r.

  Let us think about the effect of r on some coloured figures in X. The symmetry r does not fix the first coloured figure below, because it maps it to the second coloured figure, which is different. However, it does fix the third coloured figure below.

  In general, we can say the following.

  The coloured figures in X fixed by the symmetry r are those in which the bottom two outer triangles are the same colour.

For such a coloured figure there are three choices for the colour of the middle triangle, three choices for the colour of the top triangle, and three choices for the single colour of the bottom two outer triangles, so the number of such coloured figures is 3^3. Hence

$$|\mathrm{Fix}\, r| = 3^3.$$

By similar arguments,

$$|\mathrm{Fix}\, s| = 3^3$$

and

$$|\mathrm{Fix}\, t| = 3^3.$$

The sizes of the fixed sets for this group action are summarised below.

Symmetry g	$\lvert\mathrm{Fix}\, g\rvert$
e	3^4
a	3^2
b	3^2
r	3^3
s	3^3
t	3^3

In the solution to Worked Exercise E74, once we had found the size of $\mathrm{Fix}\, a$, we could see that by a similar argument we would get the same answer for the size of $\mathrm{Fix}\, b$. This is because the symmetries a and b are of the same geometric type. Similarly, once we had found the size of $\mathrm{Fix}\, r$, we could see that by similar arguments we would get the same answers for the sizes of $\mathrm{Fix}\, s$ and $\mathrm{Fix}\, t$. Again this is because the symmetries r, s and t are of the same geometric type.

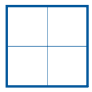
In fact, our observation that a and b are of the same geometric type is an observation that they are *conjugate* in $S(\triangle)$. You studied the connection between conjugacy and geometric type in symmetry groups in Subsection 4.1 of Unit E2: you saw there that two symmetries x and y of a figure F are conjugate in $S(F)$ if and only if there is a symmetry g of F that transforms a diagram illustrating x into a diagram illustrating y (when we ignore any labels).

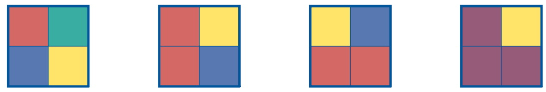
In general, if a group G of symmetries of a figure F acts in the natural way on a set X of coloured figures, then symmetries in G that are conjugate in $S(F)$ have fixed sets of the same size (but usually not the same fixed sets).

In the solution to Worked Exercise E74, the sizes of the fixed sets were left as powers of the number of colours, rather than being evaluated. You should do likewise in the next exercise, and in the subsequent exercises in this subsection. This is convenient when we use the Counting Theorem, as you will see later.

Exercise E180

Consider the action of the group $S(\square)$ (see Figure 68) on the set X whose elements are all the coloured figures obtained by colouring each of the four small squares in the figure on the left below with one of the five colours blue, yellow, red, green and purple. Some examples of elements of X are shown on the right below.





Find the size of $\operatorname{Fix} g$ for each symmetry g in $S(\square)$.

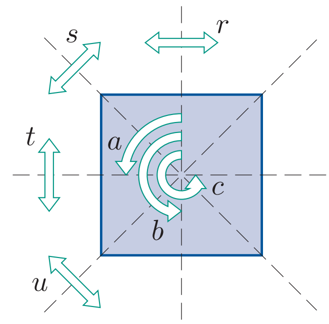


Figure 68 $S(\square)$

Exercise E181

Consider the group action that is the same as the one in Exercise E180 except that the figures in the set X are coloured with the *four* colours blue, yellow, red and green, instead of with five colours. By using your final answers to Exercise E180 and thinking about the arguments that you used to derive them, write down the size of $\operatorname{Fix} g$ for each symmetry g in $S(\square)$. You should not need to work through all the arguments again.

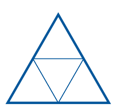
The solution to Exercise E181 illustrates that if we have found the sizes of the fixed sets for the natural action of a group of symmetries on a set of coloured figures like those in the exercise and we want to change the number of colours, then it is straightforward to find the sizes of the resulting new fixed sets.

There is a method involving permutations that can help make finding the sizes of fixed sets like those in the last few exercises and worked exercises more systematic. It is based on considering the action of the group on the *set whose elements are the parts of the figure to be coloured.* For example, in Worked Exercise E74 the parts of the figure to be coloured are the four small triangles. The mapping effect of the group $S(\triangle)$ on these four triangles is a group action by Theorem E59.

The method is demonstrated in the next worked exercise, in which we look again at the question in Worked Exercise E74, but this time use the permutation method to carry out the working.

Worked Exercise E75

Consider the action of the group $S(\triangle)$ (see Figure 69 below) on the set X whose elements are all the coloured figures obtained by colouring each of the four small triangles in the figure below with one of the three colours blue, yellow and red.



Find the size of $\operatorname{Fix} g$ for each symmetry g in $S(\triangle)$.

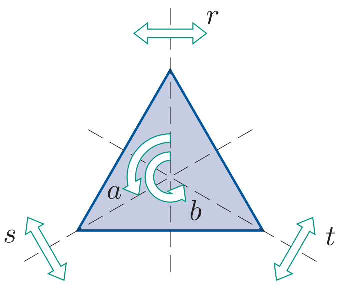
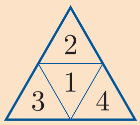


Figure 69 $S(\triangle)$

Solution

Label each part of the figure to be coloured with a symbol. Here we label the four small triangles.

We can label the figure as follows.



Express the effect of each symmetry on the parts to be coloured as a permutation, *including any 1-cycles.*

This gives the following.

Symmetry g	Permutation
e	$(1)(2)(3)(4)$
a	$(1)(2\ 3\ 4)$
b	$(1)(2\ 4\ 3)$
r	$(1)(2)(3\ 4)$
s	$(1)(3)(2\ 4)$
t	$(1)(4)(2\ 3)$

Consider the symmetry a, for example. It gives a permutation with two cycles: (1) and $(2\ 3\ 4)$. For a figure in X to be fixed by a, triangles in the same cycle must have the same colour, but triangles in different cycles can have different colours. Since there are three choices of colours for each cycle, the number of coloured figures fixed by a is 3^2.

In general, by a similar argument, if the permutation given by a symmetry g has k cycles and there are c colours (here $c = 3$), then the number of coloured figures in X fixed by g is c^k.

So we can find the sizes of the fixed sets by adding to the table as follows.

Symmetry g	Permutation	Number of cycles	$\lvert\operatorname{Fix} g\rvert$
e	$(1)(2)(3)(4)$	4	3^4
a	$(1)(2\ 3\ 4)$	2	3^2
b	$(1)(2\ 4\ 3)$	2	3^2
r	$(1)(2)(3\ 4)$	3	3^3
s	$(1)(3)(2\ 4)$	3	3^3
t	$(1)(4)(2\ 3)$	3	3^3

Notice that, as you would expect, in the solution to Worked Exercise E75 symmetries of the same geometric type give permutations with the same cycle structure, leading to fixed sets of the same sizes.

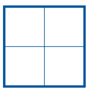
Remember that when you use the method in Worked Exercise E75 it is *essential to include 1-cycles.*

In the next exercise you are asked to answer the question in Exercise E180 again, but this time using the method in Worked Exercise E75.

Here and in similar exercises the permutations that you obtain may be different from the ones given in the solution, because there are different ways to label the parts of the figure to be coloured. However, your permutations and the ones in the solutions should have the same cycle structures and hence the same numbers of cycles.

Exercise E182

As in Exercise E180, consider the action of the group $S(\square)$ (see Figure 70) on the set X whose elements are all the coloured figures obtained by colouring each of the four small squares in the figure below with one of the five colours blue, yellow, red, green and purple.



Use the method demonstrated in Worked Exercise E75 to find the size of $\operatorname{Fix} g$ for each symmetry g in $S(\square)$.

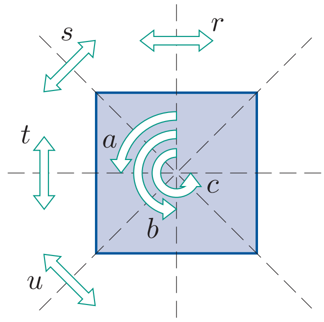


Figure 70 $S(\square)$

The permutation method introduced above will be useful in the next subsection.

4.3 The Counting Theorem and its use

In Subsection 4.1 you saw some examples of counting problems that can be interpreted as problems involving finding the number of orbits of an action of a finite group on a finite set. In this subsection you will meet the Counting Theorem and see how to use it to solve such counting problems.

The theorem is stated below. Its proof is given at the end of the subsection.

Theorem E69 Counting Theorem

Let $\wedge$ be an action of a finite group G on a finite set X. Then the number of orbits of $\wedge$ is given by

$$\frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix} g|.$$

The Counting Theorem tells us that one way to find the number of orbits of an action of a finite group G on a finite set X is to determine the number $|\operatorname{Fix} g|$ for each element g in G, add up all these numbers, and divide the total by the order of G. Here is an example.

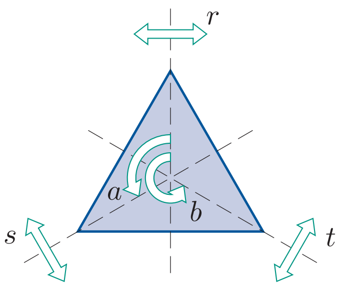
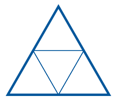


Figure 71 $S(\triangle)$

Worked Exercise E76

Use the Counting Theorem to determine how many different triangular window stickers similar to the one shown below can be made if each triangular region is to be coloured blue, yellow or red, and we regard two stickers as the same if one can be rotated or turned over to give the other.

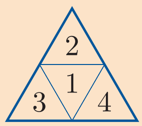


Solution

We want to regard two stickers as the same if one can be rotated or reflected to give the other, so we consider the action of the group $S(\triangle)$ (see Figure 71) on the set of all 3^4 coloured stickers in fixed positions.

The answer to the problem is the number of orbits of this group action, which we can find by using the Counting Theorem.

The sizes of the fixed sets for this group action were found in Worked Exercise E75 in the previous subsection, using the following labelled triangle.



The following table was obtained.

Symmetry g	Permutation	Number of cycles	$\lvert\text{Fix}\, g\rvert$
e	$(1)(2)(3)(4)$	4	3^4
a	$(1)(2\ 3\ 4)$	2	3^2
b	$(1)(2\ 4\ 3)$	2	3^2
r	$(1)(2)(3\ 4)$	3	3^3
s	$(1)(3)(2\ 4)$	3	3^3
t	$(1)(4)(2\ 3)$	3	3^3

By the Counting Theorem, the number of orbits is

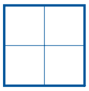
$$\begin{aligned}\tfrac{1}{6}(3^4+3^2+3^2+3^3+3^3+3^3) &= \tfrac{1}{6}(3^4+2\times 3^2+3\times 3^3)\\ &= \tfrac{1}{6}\times 3^2(3^2+2+3^2)\\ &= \tfrac{3}{2}\times 20\\ &= 30.\end{aligned}$$

Thus 30 different window stickers can be made.

So the Counting Theorem has reduced the complicated counting problem in Worked Exercise E76 to a straightforward calculation – such is the power of group theory!

Exercise E183

By using the Counting Theorem and your answers to Exercise E182 (or Exercise E180) in the previous subsection, determine how many different square headscarves, similar to the one shown below, can be made if each of the four square regions is to be coloured with one of the five colours blue, yellow, red, green and purple, and we regard two headscarves as the same if one can be rotated or turned over to give the other.



Exercise E184

Find the answer to Exercise E183 if each region of each headscarf is to be coloured with one of only *four* colours, instead of five colours.

Exercise E184 illustrates that if the number of colours in a counting problem of the type that we are considering is changed, then it is straightforward to adjust the solution accordingly.

Ferdinand Georg Frobenius

The Counting Theorem is often incorrectly referred to as *Burnside's Lemma*. The British group theorist Peter M. Neumann (1940–) explained how this name arose in his 1979 paper *A lemma that is not Burnside's*.

It appears that the result was so well known in the early twentieth century that the British mathematician William Burnside (1852–1927) quoted it without attribution in the second (1911) edition of his classic book *Theory of Groups of Finite Order*. Fifty years later, it was misattributed to Burnside by the American mathematician Solomon Golomb (1932–2016) in a paper in 1961, following which the Dutch mathematician Nicolaas Govert de Bruijn (1918–2012) referred to it as 'Burnside's lemma' in papers in 1963 and 1964. The name was used subsequently by many other mathematicians. De Bruijn wrote to Neumann as follows:

> Indeed, I think I am to blame, having used the name 'Burnside's lemma' in several of my papers. You describe correctly how this all went. Pólya did not give a reference, Golomb mentioned the name Burnside, I looked it up in Burnside's book and found it without reference, so that was that.

Augustin-Louis Cauchy

The result was known many years beforehand. It appears in a paper by the German mathematician Ferdinand Georg Frobenius (1849–1917) published in 1887, and earlier in a slightly different form in work of the French mathematician Augustin-Louis Cauchy (1789–1857) published in 1845. Neumann therefore suggested that a

more appropriate name for it is the *Cauchy–Frobenius Lemma*. This name is now sometimes used, as are some other names such as the *Counting Theorem*, but despite Neumann's paper being published only 18 years after the first misattribution of the result to Burnside in print, the name 'Burnside's lemma' is still widely used.

(Source: Neumann, P. M. (1979) 'A lemma that is not Burnside's', *Mathematical Scientist*, vol. 4, pp. 133–41.)

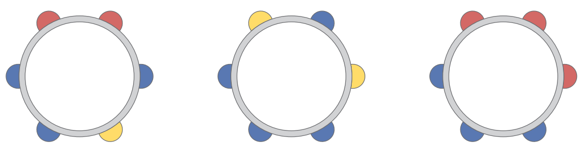
Exercise E185

Use the Counting Theorem to determine how many different 2×2 chessboards there are with each small square coloured either black or white, when we regard two chessboards as the same if one can be rotated to give the other. Hence check your answer to Exercise E175 in Subsection 4.1.

In the next worked exercise the bangle problem from Subsection 4.1 is solved using the Counting Theorem.

Worked Exercise E77

How many different bangles decorated with six equally spaced beads can be made if beads are available in blue, yellow and red? Some examples of such bangles are shown below.

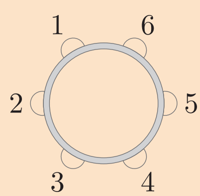


Solution

We are regarding two coloured bangles as the same if one can be rotated or turned over to give the other. So we consider the action of the group $S(\hexagon)$ on the set of all possible coloured bangles in fixed positions.

There are six beads to be chosen and three choices for the colour of each bead, so there are 3^6 coloured bangles in fixed positions.

We can label the beads as shown below.



The sizes of the fixed sets for this group action are as given below.

For convenience, here we use the permutations of the beads to represent the symmetries in $S(\hexagon)$, rather than using a different way of representing them in the first column of the table. We can do this because no two symmetries in $S(\hexagon)$ give the same permutation of the beads.

Symmetry g	Permutation	Number of cycles	$\lvert\mathrm{Fix}\, g\rvert$
e	$(1)(2)(3)(4)(5)(6)$	6	3^6
other rotations	$(1\ 2\ 3\ 4\ 5\ 6)$	1	3
	$(1\ 3\ 5)(2\ 4\ 6)$	2	3^2
	$(1\ 4)(2\ 5)(3\ 6)$	3	3^3
	$(1\ 5\ 3)(2\ 6\ 4)$	2	3^2
	$(1\ 6\ 5\ 4\ 3\ 2)$	1	3
reflections	$(1\ 6)(2\ 5)(3\ 4)$	3	3^3
	$(1\ 2)(3\ 6)(4\ 5)$	3	3^3
	$(1\ 4)(2\ 3)(5\ 6)$	3	3^3
	$(1)(4)(2\ 6)(3\ 5)$	4	3^4
	$(2)(5)(1\ 3)(4\ 6)$	4	3^4
	$(3)(6)(1\ 5)(2\ 4)$	4	3^4

By the Counting Theorem, the number of orbits is

$$\begin{aligned}
&\tfrac{1}{12}\left(3^6 + 2\times 3 + 2\times 3^2 + 4\times 3^3 + 3\times 3^4\right)\\
&= \tfrac{1}{12}\times 3\left(3^5 + 2 + 2\times 3 + 4\times 3^2 + 3\times 3^3\right)\\
&= \tfrac{1}{4}(3\times 81 + 2 + 6 + 36 + 81)\\
&= \tfrac{1}{4}(4\times 81 + 44)\\
&= 81 + 11\\
&= 92.
\end{aligned}$$

Thus there are 92 different coloured bangles.

We can reduce the amount that we have to write down in the table in the solution to Worked Exercise E77 by recognising symmetries in $S(\hexagon)$ that are of the same geometric type and hence will give permutations with the same cycle structure. This gives the following more concise table.

Symmetry g	Example permutation	Number of cycles	$\lvert\mathrm{Fix}\, g\rvert$
e	$(1)(2)(3)(4)(5)(6)$	6	3^6
2 rotations, through $\pm\pi/3$	$(1\ 2\ 3\ 4\ 5\ 6)$	1	3
2 rotations, through $\pm 2\pi/3$	$(1\ 3\ 5)(2\ 4\ 6)$	2	3^2
rotation through π	$(1\ 4)(2\ 5)(3\ 6)$	3	3^3
3 reflections not through beads	$(1\ 6)(2\ 5)(3\ 4)$	3	3^3
3 reflections through beads	$(1)(4)(2\ 6)(3\ 5)$	4	3^4

Exercise E186

Use the Counting Theorem to determine how many different bangles decorated with *five* equally spaced beads can be made, if beads are available in just *two* colours. Hence check your answer to Exercise E174(b) in Subsection 4.1.

In the next worked exercise the chessboard problem from Subsection 4.1 is solved using the Counting Theorem. The solution does not use a table of permutations to find the sizes of the fixed sets: that would be impractical, as each permutation would contain 64 symbols! Instead, it uses the type of argument that you saw in Worked Exercise E74 in the previous subsection.

Worked Exercise E78

How many different patterns can be made by colouring the squares of a chessboard either black or white?

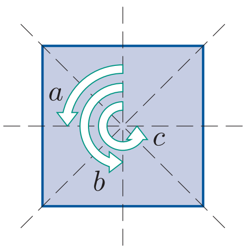


Figure 72 $S^+(\square)$

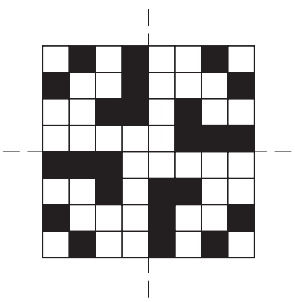


Figure 73 A coloured chessboard fixed by a

Solution

We are regarding two coloured chessboards as the same if one can be rotated to give the other. So we consider the action of the group $S^+(\square)$ (see Figure 72) on the set of all possible coloured chessboards in fixed positions.

We find the size of the fixed set of each symmetry in $S^+(\square)$ under this group action.

- First consider the identity symmetry e. It fixes all the coloured chessboards. There are 64 small squares, each coloured one of two colours, so the number of coloured chessboards is 2^{64}. Hence

  $$|\text{Fix}\, e| = 2^{64}.$$

- Now consider the symmetry a. The coloured chessboards fixed by a are those in which each square is the same colour as the three squares onto which it is mapped under successive quarter turns (an example is shown in Figure 73). There are 2^{16} different ways to colour one quarter of such a chessboard, and this colouring determines the colours of the squares in each of the other quarters of the chessboard. Thus

  $$|\text{Fix}\, a| = 2^{16}.$$

  By a similar argument,

  $$|\text{Fix}\, c| = 2^{16}.$$

- Finally consider the symmetry b. The coloured chessboards fixed by b are those in which each square is the same colour as the square onto which it is mapped under a half turn (an example is shown in Figure 74). There are 2^{32} different ways to colour one half of such a chessboard, and this colouring determines the colours of the squares in the other half. Thus

$$|\mathrm{Fix}\, b| = 2^{32}.$$

By the Counting Theorem, the number of orbits is

$$\begin{aligned}\tfrac{1}{4}\left(2^{64} + 2 \times 2^{16} + 2^{32}\right) &= \tfrac{1}{4}\left(2^{64} + 2^{17} + 2^{32}\right)\\ &= \tfrac{1}{4} \times 2^{17}\left(2^{47} + 1 + 2^{15}\right)\\ &= 2^{15}\left(2^{47} + 2^{15} + 1\right).\end{aligned}$$

This is the number of different coloured chessboards.

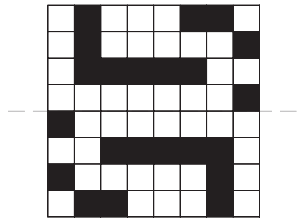


Figure 74 A coloured chessboard fixed by b

The answer found in Worked Exercise E78 is approximately 4.6×10^{18}.

Exercise E187

Use the Counting Theorem to determine how many different patterns can be made by colouring the squares of a 4×4 chessboard either black or white, when we regard two chessboards as the same if one can be obtained by rotating the other.

In the final worked exercise in this section the cube problem from Subsection 4.1 is solved using the Counting Theorem. This involves considering the action of the group $S^+(\text{cube})$ of rotations of the cube on the set of all possible coloured cubes in fixed positions. The group $S^+(\text{cube})$ has 24 elements, so it would be time-consuming to find the size of the fixed set of each of its elements individually. Instead, we collect together symmetries that are of the same geometric type and hence have fixed sets of the same size.

The solution given below includes two different versions of the part of the solution in which the sizes of the fixed sets are found. The first version uses the permutation method, and the second version does not.

Worked Exercise E79

How many different coloured cubes are there with each face painted blue, yellow or red?

Solution

We are regarding two coloured cubes as the same if one can be obtained by rotating the other. So we consider the action of the group $S^+(\text{cube})$ on the set of all possible coloured cubes in fixed positions. There are 3^6 coloured cubes in fixed positions.

The symmetries in $S^+(\text{cube})$ are of five different geometric types, as follows. Types (b)–(e) are illustrated below.

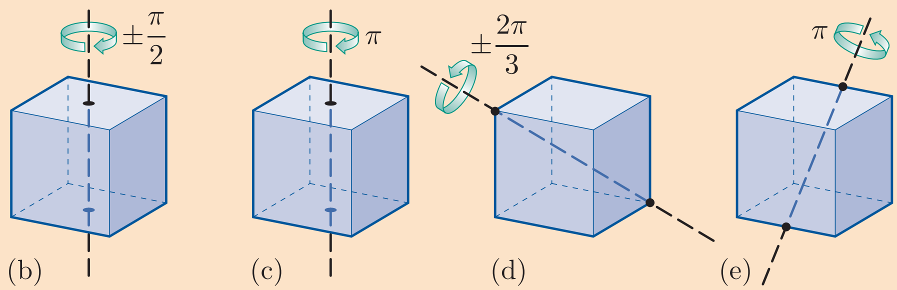
(a) The identity symmetry.

(b) Rotations through $\pm\pi/2$ about axes through midpoints of opposite faces (three such axes; two such rotations about each).

(c) Rotations through π about axes through midpoints of opposite faces (three such axes; one such rotation about each).

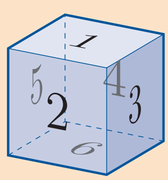
(d) Rotations through $\pm 2\pi/3$ about axes through opposite vertices (four such axes; two such rotations about each).

(e) Rotations through π about axes through midpoints of opposite edges (six such axes; one such rotation about each).



Finding the sizes of the fixed sets using the permutation method

We can label the cube as shown below.



The sizes of the fixed sets for the group action are as given below.

Symmetry g	Example permutation	Number of cycles	$\lvert\text{Fix}\, g\rvert$
identity symmetry (type (a))	(1)(2)(3)(4)(5)(6)	6	3^6
6 rotations of type (b)	(1)(2 3 4 5)(6)	3	3^3
3 rotations of type (c)	(1)(2 4)(3 5)(6)	4	3^4
8 rotations of type (d)	(1 2 5)(3 6 4)	2	3^2
6 rotations of type (e)	(1 4)(2 6)(3 5)	3	3^3

Alternative: finding the sizes of the fixed sets without using the permutation method

We consider the five different geometric types of symmetries in $S^+(\text{cube})$ in turn.

(a) **The identity symmetry (type (a))**. This fixes all the coloured cubes, so $\lvert\text{Fix}\, e\rvert = 3^6$.

(b) **Six rotations of type (b)**. Let g be such a rotation. The coloured cubes fixed by g are those in which the four faces not intersected by the axis of rotation have the same colour. So for such a cube the two faces intersected by the axis can have any colours, and the other four faces must have the same colour (as illustrated in Figure 75, with Greek letters representing the colours). Thus $\lvert\text{Fix}\, g\rvert = 3^3$.

(c) **Three rotations of type (c)**. Let g be such a rotation. The coloured cubes fixed by g are those in which each of the four faces not intersected by the axis of rotation has the same colour as its opposite face. So for such a cube the two faces intersected by the axis can have any colours, but, for the other four faces, opposite faces must have the same colour (as illustrated in Figure 76). Thus $\lvert\text{Fix}\, g\rvert = 3^4$.

(d) **Eight rotations of type (d)**. Let g be such a rotation. The coloured cubes fixed by g are those in which, for each of the two vertices on the axis of rotation, the three adjacent faces have the same colour (as illustrated in Figure 77). Thus $\lvert\text{Fix}\, g\rvert = 3^2$.

(e) **Six rotations of type (e)**. Let g be such a rotation. The coloured cubes fixed by g are those in which the two faces not touching the axis of rotation have the same colour and also, for each edge intersected by the axis of rotation, the two adjacent faces have the same colour (as illustrated in Figure 78). (The rotation g transposes the six faces in three pairs.) Thus $\lvert\text{Fix}\, g\rvert = 3^3$.

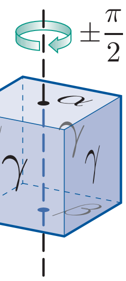


Figure 75 Colours for type (b)

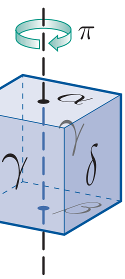


Figure 76 Colours for type (c)

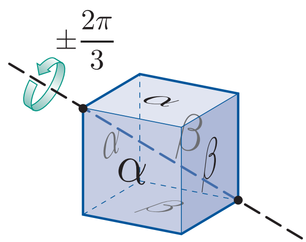


Figure 77 Colours for type (d)

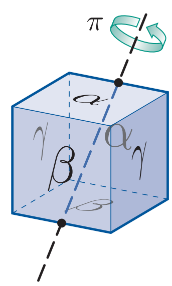


Figure 78 Colours for type (e)

Applying the Counting Theorem

By the Counting Theorem, the number of orbits is

$$\begin{aligned}
&\tfrac{1}{24}\big(3^6 + 6\times 3^3 + 3\times 3^4 + 8\times 3^2 + 6\times 3^3\big)\\
&= \tfrac{1}{24}\times 3^2\big(3^4 + 6\times 3 + 3\times 3^2 + 8 + 6\times 3\big)\\
&= \tfrac{3}{8}(81+18+27+8+18)\\
&= \tfrac{3}{8}\times 152\\
&= 3\times 19\\
&= 57.
\end{aligned}$$

Thus there are 57 different coloured cubes.

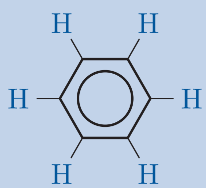
Exercise E188

How many different coloured cubes are there with each face painted blue or yellow?

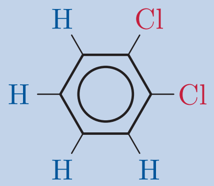
The Counting Theorem and chemical molecules

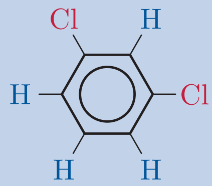
The Counting Theorem can be applied to count chemical compounds.

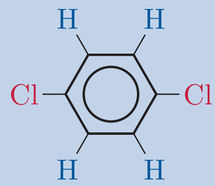
For example, *benzene* is a compound with molecular formula C_6H_6 whose molecules consist of six carbon atoms joined in a ring with a hydrogen atom attached to each, as illustrated below. (In the diagram the carbon atoms are not labelled, and the circle is a convention that indicates that the electrons forming the bonds between the carbon atoms are equally distributed.)



Chlorinated benzenes result when some of the hydrogen atoms in a benzene molecule are replaced by chlorine atoms. Replacing one hydrogen atom gives *chlorobenzene*, replacing two gives *dichlorobenzene*, replacing three gives *trichlorobenzene*, and so on. However, for some numbers of replaced hydrogen atoms there is more than one way to replace them. For example, the three different ways to replace two hydrogen atoms are shown below. These three molecules give *isomers* of dichlorobenzene. In general, isomers of a chemical compound are compounds that have the same molecular formula but different arrangements of the atoms in each molecule. They can have very different physical and chemical properties.


1,2-dichlorobenzene


1,3-dichlorobenzene


1,4-dichlorobenzene

Because the hydrogen and chlorine atoms in a chlorinated benzene molecule have a hexagonal arrangement, the problem of counting the number of different chlorinated benzenes is exactly the same as the problem of counting the number of six-bead bangles that you met earlier in this section, except with only two 'colours' (the elements hydrogen and chlorine) rather than three.

So the number of chlorinated benzenes can be worked out by changing the number of colours from three to two in the solution to Worked Exercise E77. Doing this gives the answer

$$\begin{aligned}
&\tfrac{1}{12}\left(2^6 + 2\times 2 + 2\times 2^2 + 4\times 2^3 + 3\times 2^4\right)\\
&= \tfrac{1}{12}\times 4\left(2^4 + 1 + 2 + 2^3 + 3\times 2^2\right)\\
&= \tfrac{1}{3}(16 + 1 + 2 + 8 + 12)\\
&= \tfrac{1}{3}\times 39\\
&= 13.
\end{aligned}$$

This count includes the possibility of six hydrogen atoms and no chlorine atoms, that is, benzene itself, so there are 12 different chlorinated benzenes.

Of course this count does not tell us how many isomers there are of each of dichlorobenzene, trichlorobenzene, and so on. However, there is a generalisation of the Counting Theorem known as the *Pólya Enumeration Theorem* that can be used to obtain a polynomial that provides this type of information. In the case of the chlorinated benzenes, whose underlying structure has a fairly small symmetry group, the information can be obtained quickly by drawing the different possibilities. However, for more complicated molecules there are many more possibilities and the Pólya Enumeration Theorem can provide the information much more easily.

The Pólya Enumeration Theorem was first published in 1927 by the American mathematician John Howard Redfield (1879–1944), and is sometimes known as the Redfield–Pólya Theorem. It was rediscovered independently by the Hungarian mathematician George Pólya (1887–1985). He published the result in 1937 in a paper that included the dichlorobenzene example above, as well as applications to more complicated molecules with larger symmetry groups. The paper led to an area of mathematical research known as *enumerative graph theory*.


George Pólya

Proof of the Counting Theorem

Here is a proof of the Counting Theorem.

Theorem E69 Counting Theorem

Let $\wedge$ be an action of a finite group G on a finite set X. Then the number of orbits of $\wedge$ is given by

$$\frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix} g|.$$

Proof Let the number of orbits of $\wedge$ be t.

Suppose that we find the size of $\operatorname{Stab} x$ for each element x in the set X and add up all these numbers: this gives the sum

$$\sum_{x\in X}|\operatorname{Stab} x|.$$

We will get the same answer if we split the set X into the t orbits of $\wedge$, find the value of

$$\sum_{x\in B}|\operatorname{Stab} x|$$

for each individual orbit B, and then add up these t values. Now for any orbit B of $\wedge$, we have

$$\begin{aligned}
\sum_{x\in B}|\operatorname{Stab} x| &= \sum_{x\in B}\frac{|G|}{|\operatorname{Orb} x|} \quad \text{(by the Orbit–Stabiliser Theorem)}\\
&= |G|\sum_{x\in B}\frac{1}{|\operatorname{Orb} x|}\\
&= |G|\sum_{x\in B}\frac{1}{|B|} \quad (\text{since } \operatorname{Orb} x = B \text{ for each } x\in B)\\
&= |G|\times|B|\times\frac{1}{|B|}\\
&\qquad (\text{since there are } |B| \text{ terms in the summation, each equal to } 1/|B|)\\
&= |G|.
\end{aligned}$$

Adding up this value for all t orbits gives $t|G|$, so

$$\sum_{x\in X}|\operatorname{Stab} x| = t|G|.$$

This equation can be rearranged as

$$t = \frac{1}{|G|}\sum_{x\in X}|\operatorname{Stab} x|.$$

We can now complete the proof by showing that

$$\sum_{x\in X} |\mathrm{Stab}\, x| = \sum_{g\in G} |\mathrm{Fix}\, g|.$$

To do this, consider a table whose row headings are all the elements of the group G and whose column headings are all the elements of the set X. For each g in G and each x in X such that g fixes x, we enter a tick in the cell corresponding to g and x, as illustrated below.

$$
\begin{array}{cc|ccccccccc}
 & & & & & & X & & & & \\
 & & & & & \dots & x & \dots & & & \\
\hline
 & \vdots & & & & & \vdots & & & & \\
 & g & \dots & \checkmark & \checkmark & \dots & \checkmark & \dots & \checkmark & \dots & \\
G & \vdots & & & & & \vdots & & & & \\
 & & & & & & \checkmark & & & & \\
 & & & & & & \checkmark & & & & \\
 & & & & & & \vdots & & & &
\end{array}
$$

For each x in X, the number of ticks in the column headed x is the number of elements of G that fix x, which is $|\mathrm{Stab}\, x|$. Summing over all the columns, we see that the total number of ticks in the table is

$$\sum_{x\in X} |\mathrm{Stab}\, x|.$$

But also, for each g in G, the number of ticks in the row headed g is the number of elements of X fixed by g, which is $|\mathrm{Fix}\, g|$. Summing over all the rows, we see that the total number of ticks in the table is

$$\sum_{g\in G} |\mathrm{Fix}\, g|.$$

Thus

$$\sum_{x\in X} |\mathrm{Stab}\, x| = \sum_{g\in G} |\mathrm{Fix}\, g|.$$

This completes the proof. ■

5 Group actions and groups of permutations (optional)

In this short optional section you can learn a little more about the nature of group actions, particularly those that are not *faithful*, that is, in which two or more elements of the group permute the elements of the set in the same way.

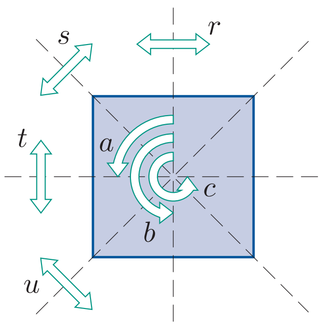


Figure 79 $S(\square)$

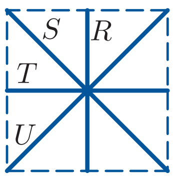


Figure 80 The lines of symmetry of the square

In Exercise E138 in Subsection 1.2 you considered the action of the group $S(\square)$ (see Figure 79) on the set $\{R, S, T, U\}$ of lines of symmetry of the square (shown on a single diagram in Figure 80). You saw that the elements of $S(\square)$ permute the lines of symmetry of the square as shown in the table below. Here i is the identity permutation of $\{R, S, T, U\}$.

Element g	Permutation
e	i
a	$(R\ T)(S\ U)$
b	i
c	$(R\ T)(S\ U)$
r	$(S\ U)$
s	$(R\ T)$
t	$(S\ U)$
u	$(R\ T)$

Thus for this group action the group $S(\square)$ splits into four subsets, namely

$$\{e, b\}, \quad \{a, c\}, \quad \{r, t\} \quad \text{and} \quad \{s, u\},$$

such that the group elements in each subset permute the elements of the set $\{R, S, T, U\}$ in the same way.

In general, as mentioned after Exercise E138, whenever a finite group G acts on a set X, the group G can be partitioned into subsets *of equal size* such that the group elements in each subset permute the elements of X in the same way. In this subsection you will meet a theorem that explains why this is, and tells you more.

First we need a preliminary theorem, Theorem E70 below. Remember that a *permutation* of a (finite or infinite) set X is a one-to-one and onto function from X to itself. We denote the set of all permutations of a set X by $\operatorname{Sym} X$. For example, for any natural number n we have $\operatorname{Sym}\{1, 2, \ldots, n\} = S_n$.

Theorem B52 in Unit B3 states that for any natural number n the set S_n of all permutations of the set $\{1, 2, \ldots, n\}$ is a group under function composition. Theorem E70 below generalises this theorem to apply to permutations of *any* set, no matter whether it is finite or infinite. Its proof is much the same as the proof of Theorem B52.

Theorem E70

Let X be any set (finite or infinite). Then the set $\operatorname{Sym} X$ of all permutations of the set X is a group under function composition.

Proof We check that the four group axioms hold for $(\operatorname{Sym} X, \circ)$ (where $\circ$ represents function composition).

G1 Closure

A composite of any two one-to-one and onto functions from X to X is a one-to-one and onto function from X to X. Thus $\operatorname{Sym} X$ is closed under function composition.

G2 Associativity

Function composition is associative.

G3 Identity

The identity function, say i, on X is an identity element for function composition in $\operatorname{Sym} X$.

G4 Inverses

Every one-to-one and onto function f from X to X has an inverse function f^{-1} that maps from X to X and satisfies $f \circ f^{-1} = i = f^{-1} \circ f$. That is, each element f of $\operatorname{Sym} X$ has an inverse f^{-1} in $\operatorname{Sym} X$ with respect to function composition.

Hence $(\operatorname{Sym} X, \circ)$ is a group. ■

For any set X, the group $\operatorname{Sym} X$ of all permutations of X is called the **symmetric group** on X. The identity element of this group, which is the identity function on X, is called the **identity permutation** of X.

We can now prove the following illuminating theorem. In the statement of this theorem the symbol $*$ is used instead of our usual symbol $\circ$ to denote the binary operation of a general group G, because the symbol $\circ$ is needed to represent function composition.

Theorem E71

Let $\wedge$ be an action of a group $(G, *)$ on a set X. For each g in G, let f_g be the permutation in $\text{Sym}\, X$ given by

$$f_g(x) = g \wedge x$$

for all $x \in X$. (That is, for each g in G the permutation f_g is the permutation of X that is the effect of g under $\wedge$.) Then the mapping

$$\begin{aligned} \phi : (G, *) &\longrightarrow (\text{Sym}\, X, \circ) \\ g &\longmapsto f_g \end{aligned}$$

is a homomorphism.

Proof Let $g, h \in G$. We have to show that

$$\phi(g * h) = \phi(g) \circ \phi(h);$$

that is,

$$f_{g*h} = f_g \circ f_h.$$

Now f_{g*h}, f_g and f_h are all functions with domain X, so to show that the equation above holds we have to show that

$$f_{g*h}(x) = (f_g \circ f_h)(x)$$

for all $x \in X$. To do this, let $x \in X$. Then

$$\begin{aligned} f_{g*h}(x) &= (g * h) \wedge x \quad \text{(by the definition of } f_{g*h}) \\ &= g \wedge (h \wedge x) \quad \text{(by axiom GA3)} \\ &= f_g(f_h(x)) \quad \text{(by the definition of } f_h \text{ and } f_g) \\ &= (f_g \circ f_h)(x) \quad \text{(by the definition of function composition).} \end{aligned}$$

Thus ϕ is a homomorphism. ■

To illustrate Theorem E71, consider once again the action of the group $S(\square)$ on the set $\{R, S, T, U\}$ of lines of symmetry of the square. As mentioned earlier in this subsection, the elements of $S(\square)$ permute the elements of $\{R, S, T, U\}$ as follows, where i is the identity permutation of X.

Element g	Permutation
e	i
a	$(R\ T)(S\ U)$
b	i
c	$(R\ T)(S\ U)$
r	$(S\ U)$
s	$(R\ T)$
t	$(S\ U)$
u	$(R\ T)$

Theorem E71 tells us that the following mapping ϕ is a homomorphism.

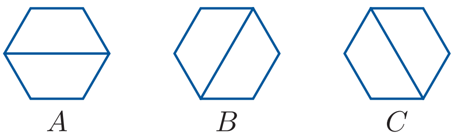
$$\begin{aligned}
\phi : (S(\square), \circ) &\longrightarrow (\mathrm{Sym}\{R, S, T, U\}, \circ)\\
e &\longmapsto i\\
a &\longmapsto (R\ T)(S\ U)\\
b &\longmapsto i\\
c &\longmapsto (R\ T)(S\ U)\\
r &\longmapsto (S\ U)\\
s &\longmapsto (R\ T)\\
t &\longmapsto (S\ U)\\
u &\longmapsto (R\ T)
\end{aligned}$$

Exercise E189

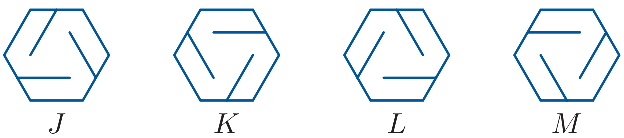
In each of parts (a) and (b) below, write down the homomorphism $\phi : (S(⬡), \circ) \longrightarrow (\mathrm{Sym}\, X, \circ)$ as defined in Theorem E71 for the action of the group $S(⬡)$ on the set X whose elements are the modified hexagons shown.

Use the labels for the elements of $S(⬡)$ shown in Figure 81.

(a)



(b)



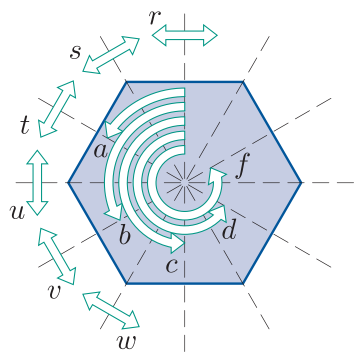


Figure 81 $S(⬡)$

We can obtain several useful results about group actions by applying the results about homomorphisms that you met in Unit E3 to the homomorphisms obtained from group actions as defined in Theorem E71. These include the following results.

Corollary E72

Let $\wedge$ be an action of a group $(G, *)$ on a set X. Then the following hold for $\wedge$.

(a) The set of permutations of X given by the elements of G under $\wedge$ is a group under function composition.

(b) The set E of elements of G that behave as the identity permutation of X is a normal subgroup of G.

(c) Two elements of G behave as the same permutation of X if and only if they lie in the same coset of E.

Proof Let $\phi : (G, *) \longrightarrow (\operatorname{Sym} X, \circ)$ be the homomorphism obtained from $\wedge$ as defined in Theorem E71.

(a) The set of permutations of X given by the elements of G under $\wedge$ is the image of ϕ. Since the image of any homomorphism is a subgroup of the codomain group (by Theorem E47 in Unit E3), this set is a group under function composition.

(b) The set E of elements of G that behave as the identity permutation of X under $\wedge$ is the kernel of ϕ. Since the kernel of any homomorphism is a normal subgroup of the domain group (by Theorem E51 in Unit E3), E is a normal subgroup of G.

(c) By Theorem E54 in Unit E3, two elements of G have the same image under ϕ if and only if they lie in the same coset of $\operatorname{Ker} \phi$ in G. That is, two elements of G behave as the same permutation of X if and only if they lie in the same coset of E. ■

By Corollary E72(b) and (c), whenever a group G acts on a set X, the group G has a normal subgroup E such that all the group elements in each coset of E behave as the same permutation of X. If G is finite, then each of these cosets contains the same number of elements, because this is always the case for cosets in a finite group. This justifies the fact mentioned near the start of this section: if a finite group G acts on a set X, then the group G can be partitioned into subsets of equal size such that the group elements in each subset permute the elements of X in the same way. (If the action of G on X is faithful, then each of the subsets contains a single element.)

Exercise E190

For each of the two group actions in Exercise E189, use your solution to Exercise E189 to partition the group $S(\hexagon)$ into subsets such that all the group elements in each subset behave as the same permutation of the set X. Write down the permutation of X corresponding to each subset.

The main theorem earlier in this section, Theorem E71, tells us that every action of a group $(G, *)$ on a set X defines a homomorphism $\phi : (G, *) \longrightarrow (\operatorname{Sym} X, \circ)$. The theorem below tells us that the converse of this theorem is also true: if $(G, *)$ is a group and X is a set then every homomorphism $\phi : (G, *) \longrightarrow (\operatorname{Sym} X, \circ)$ defines an action of $(G, *)$ on X.

You may find the expression $(\phi(g))(x)$ in the statement of this theorem rather complicated. Keep in mind that the codomain of the homomorphism ϕ is $(\operatorname{Sym} X, \circ)$, so $\phi(g)$ is a permutation of the set X and hence $(\phi(g))(x)$ is the image of x under the permutation $\phi(g)$.

Theorem E73

Let $(G, *)$ be a group, let X be a set and let $\phi : (G, *) \longrightarrow (\operatorname{Sym} X, \circ)$ be a homomorphism. Let $\wedge$ be defined by

$$g \wedge x = (\phi(g))(x)$$

for all $g \in G$ and all $x \in X$. Then $\wedge$ is an action of $(G, *)$ on X.

The next exercise asks you to prove Theorem E73. It involves working with expressions like the one mentioned above, so you may find it quite complicated.

Exercise E191

Prove Theorem E73.

Theorems E71 and E73 together show that if $(G, *)$ is a group and X is a set, then actions of $(G, *)$ on X and homomorphisms from $(G, *)$ to $(\operatorname{Sym} X, \circ)$ are essentially the same objects.

Summary

In this unit you have learned what is meant by a group action on a set, and met many examples. You have studied some general properties of group actions, and seen how some of the concepts and results that you met in earlier group theory units, such as conjugacy, Lagrange's Theorem and homomorphisms, can be viewed as particular cases of concepts and results relating to group actions. Finally, you met the *Counting Theorem* and saw how it can be used to solve counting problems that involve symmetry.

Now that you have reached the end of the group theory part of M208, you should be able to recognise how group theory reveals links and similarities in a variety of different concepts, and hence increases our understanding of them. You may be starting to appreciate the beauty and elegance of group theory as a mathematical theory in its own right, and beginning to see how it can provide powerful tools for solving some types of problems. You saw an example of its use when you solved counting problems using the Counting Theorem, but it is also used in other areas, such as cryptography, coding theory and the design of experiments. Group theory is part of the mathematical subject area known as *abstract algebra*, which is concerned with mathematical structures such as groups, fields and vector spaces.

Learning outcomes

After working through this unit, you should be able to:

- explain what is meant by a *group action*
- check the group action axioms
- explain what is meant by the *orbit* $\operatorname{Orb} x$ and the *stabiliser* $\operatorname{Stab} x$ of an element x of a set under the action of a group
- understand that the orbits of a group action form a partition of the set on which the group acts
- understand that the stabiliser of a set element under a group action is a subgroup of the group that is acting
- determine orbits and stabilisers for a group action
- understand the *Orbit–Stabiliser Theorem*
- understand various ways in which a group can act on itself or on other groups
- explain what is meant by the *fixed set* $\operatorname{Fix} g$ of an element g of a group that acts on a set
- determine fixed sets for a group action
- understand the *Counting Theorem*
- use the Counting Theorem to solve counting problems involving symmetry.

Solutions to exercises

Solution to Exercise E134

(a) **(i)** $r \wedge 2 = 3$

(ii) $b \wedge 1 = 3$

(b) **(i)** $b \wedge B = D$

(ii) $s \wedge B = B$

(c) **(i)** $(1\ 3\ 2) \wedge 2 = 1$

(ii) $(1\ 2) \wedge 3 = 3$

(d) **(i)** $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

(e) **(i)** $3 \wedge 7.4 = 3 + 7.4 = 10.4$

(ii) $1 \wedge -0.3 = 1 + (-0.3) = 0.7$

Solution to Exercise E135

We check the group action axioms.

GA1 Let $g \in G$ and let $x \in X$. Since g fixes or transposes the symbols 4 and 5, it maps each of the symbols 1, 2 and 3 to 1, 2 or 3. Therefore

$$g \wedge x = g(x) \in X.$$

Thus axiom GA1 holds.

GA2 The identity element e of G is the identity permutation of $\{1, 2, 3, 4, 5\}$. So for each x in $X = \{1, 2, 3\}$, we have

$$e \wedge x = e(x) = x.$$

Thus axiom GA2 holds.

GA3 Let $g, h \in G$ and let $x \in X$. Then

$$\begin{aligned} & g \wedge (h \wedge x) \\ &= g \wedge (h(x)) \quad \text{(by the definition of } \wedge) \\ &= g(h(x)) \quad \text{(by the definition of } \wedge) \\ &= (g \circ h)(x) \\ &\qquad \text{(by the definition of function composition)} \\ &= (g \circ h) \wedge x \quad \text{(by the definition of } \wedge). \end{aligned}$$

Thus axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

Solution to Exercise E136

(a) This mapping effect $\wedge$ does not satisfy axiom GA1 (closure). For example, $(1\ 4) \in S_5$ and $1 \in \{1, 2, 3\}$, but

$$(1\ 4) \wedge 1 = 4 \notin \{1, 2, 3\}.$$

Hence $\wedge$ is not a group action.

(This mapping effect $\wedge$ does satisfy axioms GA2 and GA3.)

(b) This mapping effect $\wedge$ does not satisfy axiom GA2 (identity).

To see this, note that the identity element of the group $(\mathbb{R}^*, \times)$ is 1, and, for example, $(4, 4) \in \mathbb{R}^2$, but

$$1 \wedge (4, 4) = (4 + 1, 4 + 1) = (5, 5) \neq (4, 4).$$

Hence $\wedge$ is not a group action.

(This mapping effect $\wedge$ does satisfy axiom GA1. However, it does not satisfy axiom GA3. To satisfy this axiom it would have to satisfy

$$g \wedge (h \wedge (x, y)) = (g \times h) \wedge (x, y)$$

for all $g, h \in \mathbb{R}^*$ and all $(x, y) \in \mathbb{R}^2$. However, for example, $1, 2 \in \mathbb{R}^*$ and $(1, 1) \in \mathbb{R}^2$, but

$$1 \wedge (2 \wedge (1, 1)) = 1 \wedge (3, 3) = (4, 4)$$

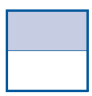
whereas

$$(1 \times 2) \wedge (1, 1) = 2 \wedge (1, 1) = (3, 3).)$$

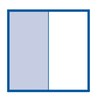
Solution to Exercise E137

(a) This is a group action.

(b) The element a of $S(\square)$ maps



to



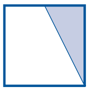
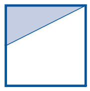
.

The first figure here is an element of X, but the second figure is not. Thus axiom GA1 does not hold. Hence $\wedge$ is not a group action.

(c) This is a group action.

(d) This is a group action.

(e) The element a of $S^+(\square)$ maps

 to  .

The first figure here is an element of X, but the second figure is not. Thus axiom GA1 does not hold. Hence $\wedge$ is not a group action.

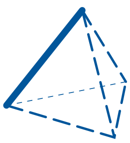
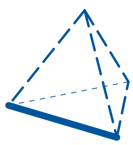
(f) This is a group action.

(g) This is a group action.

(h) This is a group action. (Each symmetry of the square maps any plane figure A in X to another plane figure, which also lies in X since X contains all plane figures.)

(i) This is a group action.

(j) By rotating the tetrahedron we can map one of the three edges in X to an edge that does not lie in X. For example, we can map

 to  .

Thus axiom GA1 does not hold. Hence $\wedge$ is not a group action.

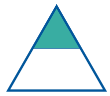
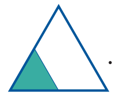
Solution to Exercise E138

The permutations are as follows. Here the identity permutation of $\{R, S, T, U\}$ is denoted by i, since e is used to denote the identity element of $S(\square)$.

Element g	Permutation
e	i
a	$(R\ T)(S\ U)$
b	i
c	$(R\ T)(S\ U)$
r	$(S\ U)$
s	$(R\ T)$
t	$(S\ U)$
u	$(R\ T)$

Solution to Exercise E139

(a) The element a of $S(\triangle)$ maps

 to  .

The first coloured figure here is an element of X, but the second is not. Thus axiom GA1 does not hold. Hence $\wedge$ is not a group action.

(b) This is a group action.

(c) This is a group action. (The set X includes every possible colour combination, so the result of applying any symmetry of the square to an element of X must be another element of X.)

Solution to Exercise E140

We check the group action axioms.

GA1 Let $g \in \mathbb{Z}$ and let $x \in \mathbb{R}$. Then

$$g \wedge x = x - g \in \mathbb{R}.$$

Thus axiom GA1 holds.

GA2 The identity element of the group $(\mathbb{Z}, +)$ is 0.

Let $x \in \mathbb{R}$. Then

$$0 \wedge x = x - 0 = x.$$

Thus axiom GA2 holds.

GA3 Let $g, h \in \mathbb{Z}$ and let $x \in \mathbb{R}$. We have to show that

$$g \wedge (h \wedge x) = (g + h) \wedge x.$$

Now

$$\begin{aligned} &g \wedge (h \wedge x) \\ &\quad = g \wedge (x - h) \quad \text{(by the definition of } \wedge) \\ &\quad = (x - h) - g \quad \text{(by the definition of } \wedge) \\ &\quad = x - (h + g) \\ &\quad = (g + h) \wedge x \quad \text{(by the definition of } \wedge). \end{aligned}$$

Thus axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

Solution to Exercise E141

This mapping effect $\wedge$ does not satisfy axiom GA2.

To see this, note that the identity element of the group $(\mathbb{Z}, +)$ is 0, and, for example, $1 \in \mathbb{R}$, but

$$0 \wedge 1 = 0 - 1 = -1 \neq 1.$$

Thus axiom GA2 does not hold.

Hence $\wedge$ is not a group action.

(This mapping effect $\wedge$ does not satisfy axiom GA3 either, but it does satisfy axiom GA1.)

Solution to Exercise E142

We show that the group action axioms hold.

GA1 Let $g \in \mathbb{R}$ and let $(x, y) \in \mathbb{R}^2$. Then

$$g \wedge (x, y) = (x, y + g) \in \mathbb{R}^2.$$

Thus axiom GA1 holds.

GA2 The identity element of the group $(\mathbb{R}, +)$ is 0.

Let $(x, y) \in \mathbb{R}^2$. Then

$$0 \wedge (x, y) = (x, y + 0) = (x, y).$$

Thus axiom GA2 holds.

GA3 Let $g, h \in \mathbb{R}$ and let $(x, y) \in \mathbb{R}^2$. We have to show that

$$g \wedge (h \wedge (x, y)) = (g + h) \wedge (x, y).$$

Now

$$\begin{aligned} g \wedge (h \wedge (x, y)) &= g \wedge (x, y + h) \\ &= (x, y + h + g) \\ &= (x, y + g + h) \end{aligned}$$

and

$$(g + h) \wedge (x, y) = (x, y + g + h).$$

The two expressions obtained are the same, so axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

Solution to Exercise E143

(a) We check the group action axioms.

GA1 The element (ax, y) is an element of $\mathbb{R}^2$ for all real numbers a, x and y, so axiom GA1 holds.

GA2 The identity element of G is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $(x, y) \in \mathbb{R}^2$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (1x, y) = (x, y).$$

So axiom GA2 holds.

GA3 Let $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \in G$ and let $(x, y) \in \mathbb{R}^2$. We have to show that

$$\begin{aligned} &\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge \left(\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \wedge (x, y) \right) \\ &= \left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \right) \wedge (x, y). \end{aligned}$$

Now

$$\begin{aligned} &\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge \left(\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \wedge (x, y) \right) \\ &= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (cx, y) \\ &= (acx, y) \end{aligned}$$

and

$$\begin{aligned} &\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \right) \wedge (x, y) \\ &= \begin{pmatrix} ac & ad + b \\ 0 & 1 \end{pmatrix} \wedge (x, y) \\ &= (acx, y). \end{aligned}$$

The two expressions obtained are the same, so axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

(b) Axiom GA2 does not hold for $\wedge$ because, for example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \wedge (2, 2) = (1 \times 2, 0 \times 2) = (2, 0) \neq (2, 2).$$

Thus $\wedge$ is not a group action.

(Axiom GA3 does not hold either, but axiom GA1 does hold.)

(c) Axiom GA3 does not hold for $\wedge$.

For axiom GA3 to hold, we require that, for all $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \in G$ and all $(x,y) \in \mathbb{R}^2$,

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge \left(\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \wedge (x,y) \right)$$
$$= \left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \right) \wedge (x,y).$$

The left-hand side of this equation is equal to

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (cx, dy+y)$$
$$= (acx, b(dy+y)+dy+y)$$
$$= (acx, bdy+by+dy+y)$$

and the right-hand side is equal to

$$\begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix} \wedge (x,y)$$
$$= (acx, (ad+b)y+y)$$
$$= (acx, ady+by+y).$$

The two expressions obtained are equal only if

$$bdy+by+dy+y = ady+by+y;$$

that is, only if

$$bdy+dy = ady,$$

which we can write as

$$(b-a+1)dy = 0.$$

This equation is not true in general. For instance, if we take $a=b=d=y=1$, it gives $1=0$.

So, for example, the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$ and the point $(0,1) \in \mathbb{R}^2$ provide a counterexample to show that axiom GA3 does not hold here.

Thus $\wedge$ is not a group action.

(Axioms GA1 and GA2 do hold.)

(It is not necessary to give the general algebraic argument above: it is sufficient just to demonstrate that there is a counterexample to axiom GA3. However, the general argument may help us find a counterexample.)

Solution to Exercise E144

The orbits are

$$\operatorname{Orb} 1 = \{1,2,3,4\},$$
$$\operatorname{Orb} 2 = \{1,2,3,4\},$$
$$\operatorname{Orb} 3 = \{1,2,3,4\},$$
$$\operatorname{Orb} 4 = \{1,2,3,4\}.$$

(So for this group action the orbit of each element is just the whole set X on which the group acts.)

Solution to Exercise E145

The orbits are

$$\operatorname{Orb} A = \{A\},$$
$$\operatorname{Orb} B = \{B,C,D\},$$
$$\operatorname{Orb} C = \{B,C,D\},$$
$$\operatorname{Orb} D = \{B,C,D\}.$$

Solution to Exercise E146

The orbits are

$$\operatorname{Orb} A = \{A,B\},$$
$$\operatorname{Orb} B = \{A,B\},$$
$$\operatorname{Orb} C = \{C,D\},$$
$$\operatorname{Orb} D = \{C,D\}.$$

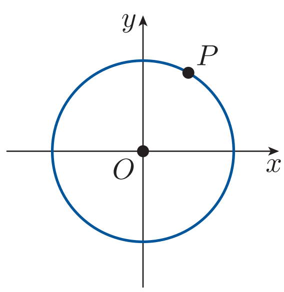
Solution to Exercise E147

The elements of the group $S(\bigcirc)$ are the rotations about O and the reflections in the lines through O.

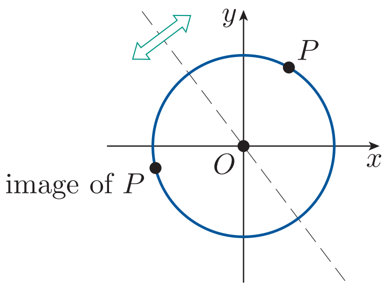
Any rotation about O and any reflection in a line through O maps O to itself, so

$$\operatorname{Orb} O = \{O\}.$$

Now let P be any other point in $\mathbb{R}^2$. The rotations in $S(\bigcirc)$ rotate P about O, through all possible angles. So $\operatorname{Orb} P$ certainly includes all points on the circle with centre O whose radius is the distance between O and P, as shown below.



Also, any reflection in a line through O maps P to a point on this circle, as illustrated below.



Hence Orb P is this circle.

(So the orbits of the points in $\mathbb{R}^2$ under the action of $S(\bigcirc)$ are the same as their orbits under the action of $S^+(\bigcirc)$, which were found in Worked Exercise E63.)

Solution to Exercise E148

The orbits are

$$\{A_1, A_3, A_7, A_9\}, \quad \{A_2, A_4, A_6, A_8\}, \quad \{A_5\}.$$

(We can find them by using Strategy E7.)

Solution to Exercise E149

(a) The orbits are

$$\{A_1, A_2, A_3, A_4\}, \quad \{A_5, A_6, A_7, A_8\}.$$

(b) There is just one orbit:

$$\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\} = X.$$

Solution to Exercise E150

(a) There is just one orbit:

$$\{1, 2, 3, 4\}.$$

(b) The orbits are

$$\{1, 3\}, \quad \{2, 4\}.$$

(c) The orbits are

$$\{1, 4\}, \quad \{2, 3\}.$$

(d) The orbits are

$$\{1\}, \quad \{2\}, \quad \{3\}, \quad \{4\}.$$

Solution to Exercise E151

In the solution to Worked Exercise E64 it was found that for any point $(x, y) \in \mathbb{R}^2$,

$$\text{Orb}(x, y) = \{(ax, by) : a, b \in \mathbb{R}^+\}.$$

(a) Putting $(x, y) = (1, 0)$ gives

$$\begin{aligned}\text{Orb}(1, 0) &= \{(a \times 1, b \times 0) : a, b \in \mathbb{R}^+\}\\ &= \{(a, 0) : a \in \mathbb{R}^+\}.\end{aligned}$$

So $\text{Orb}(1, 0)$ is the positive part of the x-axis.

(b) Putting $(x, y) = (0, -1)$ gives

$$\begin{aligned}\text{Orb}(0, -1) &= \{(a \times 0, b \times (-1)) : a, b \in \mathbb{R}^+\}\\ &= \{(0, -b) : b \in \mathbb{R}^+\}.\end{aligned}$$

So $\text{Orb}(0, -1)$ is the negative part of the y-axis.

(c) Putting $(x, y) = (1, 1)$ gives

$$\begin{aligned}\text{Orb}(1, 1) &= \{(a \times 1, b \times 1) : a, b \in \mathbb{R}^+\}\\ &= \{(a, b) : a, b \in \mathbb{R}^+\}.\end{aligned}$$

So $\text{Orb}(1, 1)$ is the first quadrant of the plane. (It does not include any points on the x-axis or y-axis.)

Solution to Exercise E152

The point $(0, 1)$ has still not been assigned to an orbit. We have

$$\begin{aligned}\text{Orb}(0, 1) &= \{(a \times 0, b \times 1) : a, b \in \mathbb{R}^+\}\\ &= \{(0, b) : b \in \mathbb{R}^+\}.\end{aligned}$$

So $\text{Orb}(0, 1)$ is the positive part of the y-axis.

The point $(-1, 1)$ has still not been assigned to an orbit. We have

$$\begin{aligned}\text{Orb}(-1, 1) &= \{(a \times (-1), b \times 1) : a, b \in \mathbb{R}^+\}\\ &= \{(-a, b) : a, b \in \mathbb{R}^+\}.\end{aligned}$$

So $\text{Orb}(-1, 1)$ is the second quadrant of the plane. (It does not include any points on the x-axis or y-axis.)

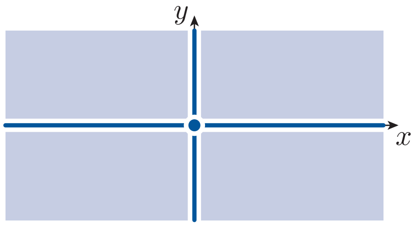
The point $(-1, -1)$ has still not been assigned to an orbit. We have

$$\begin{aligned}\text{Orb}(-1, -1) &= \{(a \times (-1), b \times (-1)) : a, b \in \mathbb{R}^+\}\\ &= \{(-a, -b) : a, b \in \mathbb{R}^+\}.\end{aligned}$$

So $\text{Orb}(-1, -1)$ is the third quadrant of the plane. (It does not include any points on the x-axis or y-axis.)

All the points in the plane have now been assigned to orbits.

The nine orbits of the group action are sketched below.



Solution to Exercise E153

For any point $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}\operatorname{Orb}(x, y) &= \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x, y) : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G\right\} \\ &= \{(ax, y) : a, b \in \mathbb{R},\ a \neq 0\} \\ &= \{(ax, y) : a \in \mathbb{R}^*\}.\end{aligned}$$

For any point of the form $(0, y)$ (that is, any point on the y-axis) we have

$$\operatorname{Orb}(0, y) = \{(a \times 0, y) : a \in \mathbb{R}^*\} = \{(0, y)\}.$$

So each point on the y-axis lies in an orbit containing itself alone. For example, $\operatorname{Orb}(0, 2) = \{(0, 2)\}$.

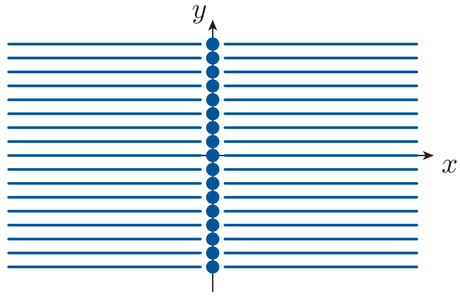
For any point of the form (x, y) where $x \neq 0$ (that is, any point not on the y-axis) we have

$$\operatorname{Orb}(x, y) = \{(ax, y) : a \in \mathbb{R}^*\}.$$

This is the set of all points on the horizontal line through the point (x, y), except for the point $(0, y)$. For example, $\operatorname{Orb}(1, 2)$ is the line $y = 2$ excluding the point $(0, 2)$.

We have now found all the orbits. They are the individual points on the y-axis and the horizontal lines excluding the point on the y-axis in each such line.

They are sketched below. Each orbit that is a line continues on the other side of the y-axis.



Solution to Exercise E154

For any point $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}\operatorname{Orb}(x, y) &= \left\{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \wedge (x, y) : \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G\right\} \\ &= \{(ax, ay) : a, b \in \mathbb{R},\ a \neq 0\} \\ &= \{(ax, ay) : a \in \mathbb{R}^*\}.\end{aligned}$$

So, for example,

$$\operatorname{Orb}(0, 0) = \{(a \times 0, a \times 0) : a \in \mathbb{R}^*\} = \{(0, 0)\}.$$

So the orbit of the point $(0, 0)$ consists of the point $(0, 0)$ alone.

Also, for example,

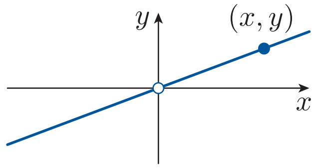
$$\begin{aligned}\operatorname{Orb}(1, 0) &= \{(a \times 1, a \times 0) : a \in \mathbb{R}^*\} \\ &= \{(a, 0) : a \in \mathbb{R}^*\}, \\ \operatorname{Orb}(0, 1) &= \{(a \times 0, a \times 1) : a \in \mathbb{R}^*\} \\ &= \{(0, a) : a \in \mathbb{R}^*\}, \\ \operatorname{Orb}(1, 2) &= \{(a \times 1, a \times 2) : a \in \mathbb{R}^*\} \\ &= \{(a, 2a) : a \in \mathbb{R}^*\}.\end{aligned}$$

So $\operatorname{Orb}(1, 0)$ consists of all the points on the x-axis excluding the origin, $\operatorname{Orb}(0, 1)$ consists of all the points on the y-axis excluding the origin, and $\operatorname{Orb}(1, 2)$ consists of all the points on the line $y = 2x$ excluding the origin.

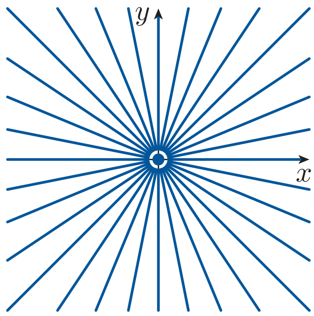
In general, as we found above, we have

$$\operatorname{Orb}(x, y) = \{(ax, ay) : a \in \mathbb{R}^*\}.$$

If x and y are not both zero, then this set consists of all the points on the line through the origin and the point (x, y), excluding the origin itself, as sketched below.



We have now found all the orbits. They are the origin, together with all the lines that pass through the origin, each excluding the origin. They are sketched below. (Each orbit that is a line continues on the other side of the origin.)



Solution to Exercise E155

The stabilisers are

$$\operatorname{Stab} 1 = \{e, s\},$$
$$\operatorname{Stab} 2 = \{e, u\},$$
$$\operatorname{Stab} 3 = \{e, s\},$$
$$\operatorname{Stab} 4 = \{e, u\}.$$

Solution to Exercise E156

The stabilisers are

$$\operatorname{Stab} A = \{e, a, b, r, s, t\} = S(\triangle),$$
$$\operatorname{Stab} B = \{e, r\},$$
$$\operatorname{Stab} C = \{e, s\},$$
$$\operatorname{Stab} D = \{e, t\}.$$

Solution to Exercise E157

The stabilisers are

$$\operatorname{Stab} A = \{e, s\},$$
$$\operatorname{Stab} B = \{e, s\},$$
$$\operatorname{Stab} C = \{e, r\},$$
$$\operatorname{Stab} D = \{e, r\}.$$

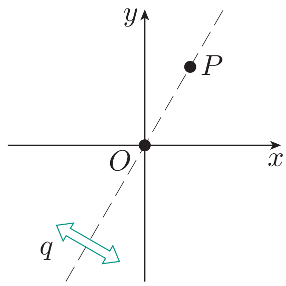
Solution to Exercise E158

The elements of the group $S(\bigcirc)$ are the rotations about O and the reflections in the lines through O.

Any rotation about O and any reflection in a line through O fixes O, so

$$\operatorname{Stab} O = S(\bigcirc).$$

Now let P be any other point in $\mathbb{R}^2$. The only rotation in $S(\bigcirc)$ that fixes P is the identity symmetry e. The only reflection in $S(\bigcirc)$ that fixes P is the reflection, say q, in the line through O and P, as illustrated below.



So

$$\operatorname{Stab} P = \{e, q\},$$

where q is the reflection in the line through O and P.

Solution to Exercise E159

The stabilisers are

$$\operatorname{Stab} A_1 = \{e, s\},$$
$$\operatorname{Stab} A_2 = \{e, r\},$$
$$\operatorname{Stab} A_3 = \{e, u\},$$
$$\operatorname{Stab} A_4 = \{e, t\},$$
$$\operatorname{Stab} A_5 = \{e, a, b, c, r, s, t, u\} = S(\square),$$
$$\operatorname{Stab} A_6 = \{e, t\},$$
$$\operatorname{Stab} A_7 = \{e, u\},$$
$$\operatorname{Stab} A_8 = \{e, r\},$$
$$\operatorname{Stab} A_9 = \{e, s\}.$$

The stabiliser $\operatorname{Stab} A_5$ is a subgroup of $S(\square)$ because it is the whole group $S(\square)$. The stabiliser of each of the other modified squares consists of the identity element e of $S(\square)$ together with an element of $S(\square)$ of order 2, so it is the subgroup of $S(\square)$ generated by that element of order 2. Thus all the stabilisers are subgroups of $S(\square)$.

Solution to Exercise E160

(a) The stabiliser of each of the modified squares is $\{e\}$, which is the trivial subgroup of $S^+(\square)$.

(b) Again, the stabiliser of each of the modified squares is the trivial subgroup $\{e\}$.

Solution to Exercise E161

From the solution to Worked Exercise E69, for any point (x, y) in $\mathbb{R}^2$,

$$\operatorname{Stab}(x,y) = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : ax = x \text{ and } by = y\right\}.$$

(a) Putting $(x, y) = (2, 0)$ gives

$$\begin{aligned}
&\operatorname{Stab}(2,0)\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a \times 2 = 2 \text{ and } b \times 0 = 0\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : 2a = 2 \text{ and } 0 = 0\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a = 1\right\}\\
&= \left\{\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} : b \in \mathbb{R}^+\right\}.
\end{aligned}$$

(This is the same subgroup of $(G, \times)$ as $\operatorname{Stab}(-1, 0)$, found in Worked Exercise E69(b).)

(b) Putting $(x, y) = (0, 5)$ gives

$$\begin{aligned}
&\operatorname{Stab}(0,5)\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a \times 0 = 0 \text{ and } b \times 5 = 5\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : 0 = 0 \text{ and } 5b = 5\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : b = 1\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^+\right\}.
\end{aligned}$$

Solution to Exercise E162

From the solution to Worked Exercise E69, for any point $(x, y) \in \mathbb{R}^2$,

$$\operatorname{Stab}(x,y) = \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : ax = x \text{ and } by = y\right\}.$$

(a) Consider a point of the form $(x, 0)$ where $x \in \mathbb{R}^*$. By the expression for $\operatorname{Stab}(x, y)$ above, we have

$$\begin{aligned}
&\operatorname{Stab}(x,0)\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : ax = x \text{ and } b \times 0 = 0\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : ax = x\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a = 1\right\} \quad (\text{since } x \neq 0)\\
&= \left\{\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} : b \in \mathbb{R}^+\right\}.
\end{aligned}$$

This is the same subgroup as found in Exercise E161(a).

(b) Consider a point of the form $(0, y)$ where $y \in \mathbb{R}^*$. By the expression for $\operatorname{Stab}(x, y)$ above, we have

$$\begin{aligned}
&\operatorname{Stab}(0,y)\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a \times 0 = 0 \text{ and } by = y\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : by = y\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : b = 1\right\} \quad (\text{since } y \neq 0)\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^+\right\}.
\end{aligned}$$

This is the same subgroup as found in Exercise E161(b).

(c) Consider a point of the form (x, y) where $x, y \in \mathbb{R}^*$. By the expression for $\operatorname{Stab}(x, y)$ above, we have

$$\begin{aligned}
&\operatorname{Stab}(x,y)\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : ax = x \text{ and } by = y\right\}\\
&= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G : a = 1 \text{ and } b = 1\right\}\\
&\qquad\qquad\qquad (\text{since } x, y \neq 0)\\
&= \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}.
\end{aligned}$$

This is the trivial subgroup of $(G, \times)$.

Solution to Exercise E163

For any point $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
&\operatorname{Stab}(x, y)\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x, y) = (x, y) \right\}\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G : (ax, y) = (x, y) \right\}\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G : ax = x \text{ and } y = y \right\}\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G : ax = x \right\}.
\end{aligned}$$

Since we can simplify the equation $ax = x$ in the expression above if we know that $x \neq 0$, we now consider the cases $x \neq 0$ and $x = 0$ separately.

For any point $(x, y) \in \mathbb{R}^2$ with $x \neq 0$ (that is, any point not on the y-axis), we have

$$\begin{aligned}
\operatorname{Stab}(x, y) &= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G : ax = x \right\}\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G : a = 1 \right\}\\
&\qquad\qquad\qquad\qquad (\text{since } x \neq 0)\\
&= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.
\end{aligned}$$

For any point $(0, y) \in \mathbb{R}^2$ (that is, any point on the y-axis), we have

$$\begin{aligned}
\operatorname{Stab}(0, y) &= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G : a \times 0 = 0 \right\}\\
&= G.
\end{aligned}$$

In summary, the stabiliser of any point on the y-axis is the whole group G, and the stabiliser of any other point is the subgroup

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$

Solution to Exercise E164

For any point $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
&\operatorname{Stab}(x, y)\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G : \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \wedge (x, y) = (x, y) \right\}\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G : (ax, ay) = (x, y) \right\}\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G : ax = x \text{ and } ay = y \right\}.
\end{aligned}$$

Now if we know that $x \neq 0$ then we can simplify the equation $ax = x$ to $a = 1$. Similarly, if we know that $y \neq 0$ then we can simplify the equation $ay = y$ to $a = 1$. So we now split into two cases: the case where *either* $x \neq 0$ *or* $y \neq 0$ (or both), that is, the case where x and y are not both zero, and the remaining case, which is $x = y = 0$.

For any point $(x, y) \in \mathbb{R}^2$ such that $x \neq 0$ or $y \neq 0$ (that is, any point except the origin), we have

$$\begin{aligned}
\operatorname{Stab}(x, y) &= \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G : a = 1 \right\}\\
&= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.
\end{aligned}$$

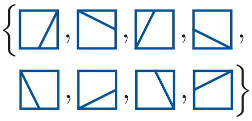
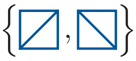
The only remaining point is the origin, for which we have

$$\begin{aligned}
&\operatorname{Stab}(0, 0)\\
&= \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in G : a \times 0 = 0 \text{ and } a \times 0 = 0 \right\}\\
&= G.
\end{aligned}$$

In summary, the stabiliser of the origin is the whole group G, and the stabiliser of any other point is the subgroup

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$

Solution to Exercise E165

A	Orb A	Stab A	$\lvert\text{Orb}\,A\rvert$	$\lvert\text{Stab}\,A\rvert$
		$\{e, b\}$	4	2
		$\{e\}$	8	1
		$\{e, b, s, u\}$	2	4
		$S(\square)$	1	8

In each case, the number of elements in the orbit multiplied by the number of elements in the stabiliser is 8, the order of the group $S(\square)$.

Solution to Exercise E166

(a) This group action has just one orbit, namely

$$\{1, 2, 3, 4\}.$$

Also,

$$\begin{aligned}\text{Stab}\,1 &= \text{Stab}\,3 = \{e, s\},\\ \text{Stab}\,2 &= \text{Stab}\,4 = \{e, u\}.\end{aligned}$$

Hence, for each vertex label x,

$$|\text{Orb}\,x| \times |\text{Stab}\,x| = 4 \times 2 = 8 = |S(\square)|.$$

(b) The orbits of this group action are

$$\{R, T\}, \quad \{S, U\}.$$

Also,

$$\begin{aligned}\text{Stab}\,R &= \text{Stab}\,T = \{e, b, r, t\},\\ \text{Stab}\,S &= \text{Stab}\,U = \{e, b, s, u\}.\end{aligned}$$

Hence, for each line of symmetry x,

$$|\text{Orb}\,x| \times |\text{Stab}\,x| = 2 \times 4 = 8 = |S(\square)|.$$

(c) The orbits of this group action are

$$\{A\}, \quad \{B, C, D\}.$$

Also,

$$\begin{aligned}\text{Stab}\,A &= S(\triangle),\\ \text{Stab}\,B &= \{e, r\},\\ \text{Stab}\,C &= \{e, s\},\\ \text{Stab}\,D &= \{e, t\}.\end{aligned}$$

So for the modified triangle A we have

$$|\text{Orb}\,A| \times |\text{Stab}\,A| = 1 \times 6 = 6 = |S(\triangle)|.$$

For each other modified triangle, say x, we have

$$|\text{Orb}\,x| \times |\text{Stab}\,x| = 3 \times 2 = 6 = |S(\triangle)|.$$

(The orbits and stabilisers under the three group actions in this exercise were found in the solutions to worked exercises and exercises in Subsections 2.1 and 2.3, but it is probably quicker to find them again than to look back.)

Solution to Exercise E167

(a) $\text{Stab}\,2 = \{e, u\}$.

(b) The left cosets of $\text{Stab}\,2$ in $S(\square)$ are

$$\begin{aligned}\text{Stab}\,2 &= \{e, u\},\\ a\,\text{Stab}\,2 &= \{a \circ e, a \circ u\} = \{a, r\},\\ b\,\text{Stab}\,2 &= \{b \circ e, b \circ u\} = \{b, s\},\\ c\,\text{Stab}\,2 &= \{c \circ e, c \circ u\} = \{c, t\}.\end{aligned}$$

(c) We can see from Figure 49 that

e and u map 2 to 2,
a and r map 2 to 3,
b and s map 2 to 4,
c and t map 2 to 1.

So the partition of $S(\square)$ according to where its elements map 2 is

$$\{e, u\}, \quad \{a, r\}, \quad \{b, s\}, \quad \{c, t\}.$$

(d) The partitions found in parts (b) and (c) are the same.

Solution to Exercise E168

(a) The elements of S_3 that fix 1 are e and $(2\ 3)$, so

$$\text{Stab}\,1 = \{e,\ (2\ 3)\}.$$

(b) The left cosets of Stab 1 in S_3 are

$$\begin{aligned}
\operatorname{Stab}1 &= \{e,\ (2\ 3)\},\\
(1\ 2)\operatorname{Stab}1 &= \{(1\ 2)\circ e,\ (1\ 2)\circ(2\ 3)\}\\
&= \{(1\ 2),\ (1\ 2\ 3)\},\\
(1\ 3)\operatorname{Stab}1 &= \{(1\ 3)\circ e,\ (1\ 3)\circ(2\ 3)\}\\
&= \{(1\ 3),\ (1\ 3\ 2)\}.
\end{aligned}$$

(c) We have

e and $(2\ 3)$ map 1 to 1,
$(1\ 2)$ and $(1\ 2\ 3)$ map 1 to 2,
$(1\ 3)$ and $(1\ 3\ 2)$ map 1 to 3.

So the partition of S_3 according to where its elements map the symbol 1 is

$$\{e,\ (2\ 3)\},\quad \{(1\ 2),\ (1\ 2\ 3)\},\quad \{(1\ 3),\ (1\ 3\ 2)\}.$$

(d) The partitions found in parts (b) and (c) are the same, as expected.

Solution to Exercise E169

The mapping f obtained from Stab 2 is

$$\begin{aligned}
f : \text{set of left cosets of Stab}\,2 &\longrightarrow \operatorname{Orb}2\\
\{e,u\} &\longmapsto 2\\
\{a,r\} &\longmapsto 3\\
\{b,s\} &\longmapsto 4\\
\{c,t\} &\longmapsto 1.
\end{aligned}$$

Solution to Exercise E170

(a) This is a group action. We show that the group action axioms hold.

GA1 Let $g, x \in G$. Then

$$g \wedge x = gx \in G.$$

Thus axiom GA1 holds.

GA2 Let e be the identity element of G and let $x \in G$. Then

$$e \wedge x = ex = x.$$

Thus axiom GA2 holds.

GA3 Let $g, h, x \in G$. Then

$$\begin{aligned}
g \wedge (h \wedge x) &= g \wedge hx\\
&= ghx\\
&= (gh) \wedge x.
\end{aligned}$$

Thus axiom GA3 holds.

Hence $\wedge$ is a group action.

(Some texts refer to this group action as the *left regular action* of a group. The proof of Cayley's Theorem given in Section 6 of Unit B3 amounts to showing that a finite group is isomorphic to the group formed by the permutations that are the effects of its elements under its left regular action. The fact that these permutations form a group follows from a result given in the optional Section 5 at the end of this unit.)

(b) This is not a group action. Axiom GA3 does not hold. If $g, h, x \in G$, then

$$g \wedge (h \wedge x) = g \wedge (xh) = xhg$$

but

$$(gh) \wedge x = xgh.$$

These two expressions are equal when $gh = hg$. This is not true in general, but it does hold when the group G is abelian.

As a particular counterexample to demonstrate that axiom GA3 does not hold, consider the group $S(\square)$ and its elements a, r and e. We have

$$a \wedge (r \wedge e) = a \wedge (e \circ r) = a \wedge r = r \circ a = u$$

but

$$(a \circ r) \wedge e = s \wedge e = e \circ s = s.$$

(c) This is a group action. We show that the group action axioms hold.

GA1 Let $g, x \in G$. Then

$$g \wedge x = xg^{-1} \in G.$$

Thus axiom GA1 holds.

GA2 Let e be the identity element of G and let $x \in G$. Then

$$e \wedge x = xe^{-1} = xe = x.$$

Thus axiom GA2 holds.

GA3 Let $g, h, x \in G$. Then

$$\begin{aligned}
g \wedge (h \wedge x) &= g \wedge (xh^{-1})\\
&= xh^{-1}g^{-1}\\
&= x(gh)^{-1}\\
&= (gh) \wedge x.
\end{aligned}$$

Thus axiom GA3 holds.

Hence $\wedge$ is a group action.

(Some texts refer to this group action as the *right regular action* of a group.)

Solution to Exercise E171

We show that the group action axioms hold.

GA1 Let $h \in H$ and let $g \in G$. Then

$$h \wedge g = hg \in G.$$

Thus axiom GA1 holds.

GA2 Let e be the identity element of H and let $g \in G$. The identity element of H is the same as the identity element of G, so

$$e \wedge g = eg = g.$$

Thus axiom GA2 holds.

GA3 Let $h_1, h_2 \in H$ and let $g \in G$. Then

$$\begin{aligned} h_1 \wedge (h_2 \wedge g) &= h_1 \wedge (h_2 g) \\ &= h_1 h_2 g \\ &= (h_1 h_2) \wedge g. \end{aligned}$$

Thus axiom GA3 holds.

Hence $\wedge$ is a group action.

Solution to Exercise E172

We show that the group action axioms hold.

GA1 Let $g \in G$ and let $h \in H$. Then

$$g \wedge h = \phi(g) * h,$$

which is in H, since $\phi(g) \in H$. Thus axiom GA1 holds.

GA2 Let $h \in H$, and let e_G and e_H be the identity elements of $(G, \circ)$ and $(H, *)$, respectively. Then

$$\begin{aligned} e_G \wedge h &= \phi(e_G) * h \\ &= e_H * h \\ &\quad (\text{since } \phi(e_G) = e_H, \text{ because } \phi \text{ is a homomorphism}) \\ &= h. \end{aligned}$$

Thus axiom GA2 holds.

GA3 Let $g_1, g_2 \in G$ and let $h \in H$. Then

$$\begin{aligned} & g_1 \wedge (g_2 \wedge h) \\ &= \phi(g_1) \wedge (\phi(g_2) * h) \quad (\text{by the definition of } \wedge) \\ &= \phi(g_1) * (\phi(g_2) * h) \quad (\text{by the definition of } \wedge) \\ &= (\phi(g_1) * \phi(g_2)) * h \quad (\text{by associativity in } (H, *)) \\ &= \phi(g_1 \circ g_2) * h \quad (\text{since } \phi \text{ is a homomorphism}) \\ &= (g_1 \circ g_2) \wedge h \quad (\text{by the definition of } \wedge). \end{aligned}$$

Thus axiom GA3 holds.

Hence $\wedge$ is a group action.

Solution to Exercise E173

There are four positions to be filled by a tile of a chosen colour, and there is a choice of five colours for each position. Hence by the Multiplication Principle the number of different patterns is

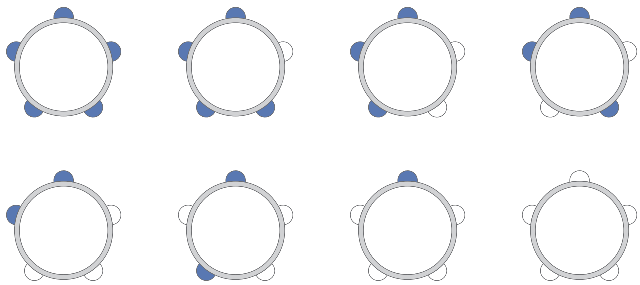
$$5 \times 5 \times 5 \times 5 = 5^4 = 625.$$

Solution to Exercise E174

(a) By the Multiplication Principle, the number of different bangles in fixed positions is

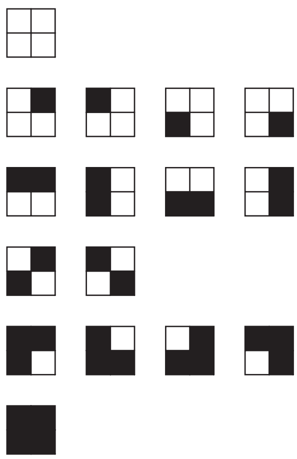
$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32.$$

(b) If two bangles are regarded as the same whenever one can be rotated or turned over to give the other, then there are eight different bangles, as shown below.



Solution to Exercise E175

(a) The sixteen coloured chessboards are drawn below. Those that can be rotated to give each other are drawn in the same row.



(b) If we regard two coloured chessboards as the same when one can be rotated to give the other, then there are six different coloured chessboards, namely those in the first column above.

Solution to Exercise E176

The fixed sets are

$$\begin{aligned}
\operatorname{Fix} e &= \{1,2,3,4\},\\
\operatorname{Fix} a &= \varnothing,\\
\operatorname{Fix} b &= \varnothing,\\
\operatorname{Fix} c &= \varnothing,\\
\operatorname{Fix} r &= \varnothing,\\
\operatorname{Fix} s &= \{1,3\},\\
\operatorname{Fix} t &= \varnothing,\\
\operatorname{Fix} u &= \{2,4\}.
\end{aligned}$$

Solution to Exercise E177

The fixed sets are

$$\begin{aligned}
\operatorname{Fix} e &= \{A,B,C,D\},\\
\operatorname{Fix} a &= \{A\},\\
\operatorname{Fix} b &= \{A\},\\
\operatorname{Fix} r &= \{A,B\},\\
\operatorname{Fix} s &= \{A,C\},\\
\operatorname{Fix} t &= \{A,D\}.
\end{aligned}$$

Solution to Exercise E178

The fixed sets are

$$\begin{aligned}
\operatorname{Fix} e &= \{A,B,C,D\},\\
\operatorname{Fix} a &= \varnothing,\\
\operatorname{Fix} r &= \{C,D\},\\
\operatorname{Fix} s &= \{A,B\}.
\end{aligned}$$

Solution to Exercise E179

(a) For any matrix $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in G$ (so $a, b \in \mathbb{R}$, $a \neq 0$),

$$\begin{aligned}
&\operatorname{Fix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\\
&= \left\{(x,y) \in \mathbb{R}^2 : \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \wedge (x,y) = (x,y)\right\}\\
&= \{(x,y) \in \mathbb{R}^2 : (ax, y) = (x,y)\}\\
&= \{(x,y) \in \mathbb{R}^2 : ax = x \text{ and } y = y\}\\
&= \{(x,y) \in \mathbb{R}^2 : ax = x\}.
\end{aligned}$$

(b) **(i)** By part (a),

$$\begin{aligned}
\operatorname{Fix} \begin{pmatrix} -1 & 5 \\ 0 & 1 \end{pmatrix} &= \{(x,y) \in \mathbb{R}^2 : -1x = x\}\\
&= \{(x,y) \in \mathbb{R}^2 : x = 0\}\\
&= \{(0,y) : y \in \mathbb{R}\}.
\end{aligned}$$

So this fixed set is the y-axis.

(ii) By part (a),

$$\begin{aligned}
\operatorname{Fix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} &= \{(x,y) \in \mathbb{R}^2 : 1x = x\}\\
&= \mathbb{R}^2.
\end{aligned}$$

So this fixed set is the whole plane $\mathbb{R}^2$.

Solution to Exercise E180

- First consider the identity symmetry e in $S(\square)$. It fixes all the coloured figures in X. There are four small squares, each coloured with one of five colours, so the number of coloured figures in X is 5^4. Hence

  $$|\operatorname{Fix} e| = 5^4.$$

- Now consider the symmetry a. The coloured figures in X fixed by a are those in which the four small squares are all the same colour. Hence

  $$|\operatorname{Fix} a| = 5.$$

  By a similar argument,

  $$|\operatorname{Fix} c| = 5.$$

- Next consider the symmetry b. The coloured figures in X fixed by b are those in which each small square is the same colour as the diagonally opposite small square. There are two pairs of diagonally opposite small squares and five choices for the colour of each pair, so

  $$|\operatorname{Fix} b| = 5^2.$$

- Next consider the symmetry r. The coloured figures in X fixed by r are those in which each small square is the same colour as the square next to it horizontally. There are two pairs of small squares next to each other horizontally and five choices for the colour of each pair, so

  $$|\operatorname{Fix} r| = 5^2.$$

  By a similar argument,

  $$|\operatorname{Fix} t| = 5^2.$$

- Next consider the symmetry s. The coloured figures in X fixed by s are those in which the top right small square is the same colour as the bottom left small square. Hence there are three colour choices to be made – the single colour of the top right and bottom left small squares, and the colour of each of the other two small squares. Each colour choice is from five colours, so

  $$|\mathrm{Fix}\, s| = 5^3.$$

  By a similar argument,

  $$|\mathrm{Fix}\, u| = 5^3.$$

The sizes of the fixed sets for this group action are summarised below.

Symmetry g	$\lvert\mathrm{Fix}\, g\rvert$
e	5^4
a	5
b	5^2
c	5
r	5^2
s	5^3
t	5^2
u	5^3

(Notice that, as expected, symmetries that are conjugate in $S(\square)$ have fixed sets of the same size. The conjugacy classes of $S(\square)$, found in Worked Exercise E28 in Subsection 2.3 of Unit E2, are

$$\{e\}, \quad \{a, c\}, \quad \{b\}, \quad \{r, t\}, \quad \{s, u\}.)$$

Solution to Exercise E181

Each of the fixed sets in Exercise E180 has size c^k where c is the number of colours and k is the number of colour choices to be made. If we change the number of colours from 5 to 4, then a fixed set of size 5^k changes to a fixed set of size 4^k. So the sizes of the fixed sets for the group action with four colours are as given below.

Symmetry g	$\lvert\mathrm{Fix}\, g\rvert$
e	4^4
a	4
b	4^2
c	4
r	4^2
s	4^3
t	4^2
u	4^3

Solution to Exercise E182

We can label the figure as follows.

2	1
3	4

This gives the following.

Symmetry g	Permutation	Number of cycles	$\lvert\mathrm{Fix}\, g\rvert$
e	$(1)(2)(3)(4)$	4	5^4
a	$(1\ 2\ 3\ 4)$	1	5
b	$(1\ 3)(2\ 4)$	2	5^2
c	$(1\ 4\ 3\ 2)$	1	5
r	$(1\ 2)(3\ 4)$	2	5^2
s	$(1\ 3)(2)(4)$	3	5^3
t	$(1\ 4)(2\ 3)$	2	5^2
u	$(1)(3)(2\ 4)$	3	5^3

(Your permutations may be different if you chose a different labelling of the squares.)

Solution to Exercise E183

We are regarding two coloured headscarves as the same if one can be rotated or reflected to give the other. So we consider the action of the group $S(\square)$ on the set of all possible coloured headscarves in fixed positions. The sizes of the fixed sets for this group action, found in Exercise E182 (and Exercise E180) in the previous subsection, are shown below.

Symmetry g	$\lvert\text{Fix}\, g\rvert$
e	5^4
a	5
b	5^2
c	5
r	5^2
s	5^3
t	5^2
u	5^3

By the Counting Theorem, the number of orbits is

$$\begin{aligned}
&\tfrac{1}{8}\left(5^4+5+5^2+5+5^2+5^3+5^2+5^3\right)\\
&=\tfrac{1}{8}\left(5^4+2\times 5+3\times 5^2+2\times 5^3\right)\\
&=\tfrac{5}{8}\left(5^3+2+3\times 5+2\times 5^2\right)\\
&=\tfrac{5}{8}(125+2+15+50)\\
&=\tfrac{5}{8}\times 192\\
&=120.
\end{aligned}$$

Thus 120 different headscarves can be made.

Solution to Exercise E184

Consider the action of the group $S(\square)$ on the set of all possible coloured headscarves in fixed positions. The sizes of the fixed sets for this group action are the same as those given in the solution to Exercise E183, but with 4 colours replacing 5 colours (as you saw in Exercise E181 in the previous subsection).

Thus the sizes of the fixed sets are as follows.

Symmetry g	$\lvert\text{Fix}\, g\rvert$
e	4^4
a	4
b	4^2
c	4
r	4^2
s	4^3
t	4^2
u	4^3

By the Counting Theorem, the number of orbits is

$$\begin{aligned}
&\tfrac{1}{8}\left(4^4+4+4^2+4+4^2+4^3+4^2+4^3\right)\\
&=\tfrac{1}{8}\left(4^4+2\times 4+3\times 4^2+2\times 4^3\right)\\
&=\tfrac{1}{2}\left(4^3+2+3\times 4+2\times 4^2\right)\\
&=2\times 4^2+1+3\times 2+4^2\\
&=32+1+6+16\\
&=55.
\end{aligned}$$

Thus 55 different headscarves can be made if only four colours are allowed.

Solution to Exercise E185

We are regarding two coloured chessboards as the same if one can be rotated to give the other. So we consider the action of the group $S^+(\square)$ on the set of all 2^4 coloured 2×2 chessboards in fixed positions.

We can label the squares of the chessboard as follows.

2	1
3	4

Thus the sizes of the fixed sets are as follows.

Symmetry g	Permutation	Number of cycles	$\lvert\text{Fix}\, g\rvert$
e	$(1)(2)(3)(4)$	4	2^4
a	$(1\ 2\ 3\ 4)$	1	2
b	$(1\ 3)(2\ 4)$	2	2^2
c	$(1\ 4\ 3\ 2)$	1	2

(Your permutations may be different if you chose a different labelling of the squares.)

(The sizes of the fixed sets are the same as the first four sizes of fixed sets given in the solutions to Exercises E183 and E184, but with two colours replacing five or four colours, respectively. There are only four fixed sets to be considered here, rather than eight, because we are considering the action of the group $S^+(\square)$ rather than that of the whole symmetry group $S(\square)$.)

By the Counting Theorem, the number of orbits is

$$\begin{aligned}
\tfrac{1}{4}\left(2^4+2+2^2+2\right)&=\tfrac{1}{4}(16+2+4+2)\\
&=\tfrac{1}{4}\times 24\\
&=6.
\end{aligned}$$

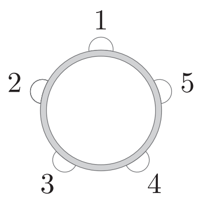
Thus there are 6 different coloured chessboards.

This is the same answer as found in Exercise E175.

Solution to Exercise E186

We are regarding two coloured bangles as the same if one can be rotated or turned over to give the other. So we consider the action of the group $S(\pentagon)$ on the set of all possible coloured bangles in fixed positions. As there are two colours, there are 2^5 coloured bangles in fixed positions.

We can label the beads as shown below.



The sizes of the fixed sets for this group action are as given below. (For convenience, we collect together symmetries of the same geometric type.)

Symmetry g	Example permutation	Number of cycles	$\lvert\text{Fix } g\rvert$
e	$(1)(2)(3)(4)(5)$	5	2^5
2 rotations, through $\pm 2\pi/5$	$(1\ 2\ 3\ 4\ 5)$	1	2
2 rotations, through $\pm 4\pi/5$	$(1\ 3\ 5\ 2\ 4)$	1	2
5 reflections	$(1)(2\ 5)(3\ 4)$	3	2^3

By the Counting Theorem, the number of orbits is

$$\begin{aligned}&\tfrac{1}{10}\left(2^5 + 2\times 2 + 2\times 2 + 5\times 2^3\right)\\ &= \tfrac{1}{10}(32+4+4+40)\\ &= \tfrac{1}{10}\times 80\\ &= 8.\end{aligned}$$

Thus eight different coloured bangles can be made.

This is the same answer as found in Exercise E174(b).

Solution to Exercise E187

We consider the action of the group $S^+(\square) = \{e, a, b, c\}$ on the set of all possible coloured chessboards in fixed positions.

- The identity symmetry e fixes all the coloured chessboards. There are 16 small squares, each coloured one of two colours, so the number of coloured chessboards is 2^{16}. Hence

  $$\lvert\text{Fix } e\rvert = 2^{16}.$$

- Now consider the symmetry a. The coloured chessboards fixed by a are those in which each square is the same colour as the three squares onto which it is mapped under successive quarter turns. There are 2^4 different ways to colour one quarter of such a chessboard, and this colouring determines the colours of the squares in each of the other quarters of the chessboard. Thus

  $$\lvert\text{Fix } a\rvert = 2^4.$$

  By a similar argument,

  $$\lvert\text{Fix } c\rvert = 2^4.$$

- Finally consider the symmetry b. The coloured chessboards fixed by b are those in which each square is the same colour as the square onto which it is mapped under a half turn. There are 2^8 different ways to colour one half of such a chessboard, and this colouring determines the colours of the squares in the other half. Thus

  $$\lvert\text{Fix } b\rvert = 2^8.$$

By the Counting Theorem, the number of orbits is

$$\begin{aligned}\tfrac{1}{4}\left(2^{16} + 2\times 2^4 + 2^8\right) &= \tfrac{1}{4}\left(2^{16} + 2^5 + 2^8\right)\\ &= \tfrac{1}{4}\times 2^5\left(2^{11} + 1 + 2^3\right)\\ &= 2^3\left(2^{11} + 2^3 + 1\right).\end{aligned}$$

This is the number of different coloured chessboards.

(The number $2^3\left(2^{11}+2^3+1\right)$ is a little time-consuming to evaluate without a calculator. A calculator shows that its value is 16 456.)

Solution to Exercise E188

To determine the number of coloured cubes when only two colours are available, we rework the calculation in the solution to Worked Exercise E79, changing the number of colours from 3 to 2 wherever it occurs. This gives the answer

$$\begin{aligned}
&\tfrac{1}{24}\left(2^6 + 6\times 2^3 + 3\times 2^4 + 8\times 2^2 + 6\times 2^3\right)\\
&= \tfrac{1}{24}\times 2^2\left(2^4 + 6\times 2 + 3\times 2^2 + 8 + 6\times 2\right)\\
&= \tfrac{1}{6}(16 + 12 + 12 + 8 + 12)\\
&= \tfrac{1}{6}\times 60\\
&= 10.
\end{aligned}$$

So there are 10 different coloured cubes if only two colours are available.

Solution to Exercise E189

(a) The homomorphism ϕ is as follows, where e and i are the identity elements of $(S(\hexagon),\circ)$ and $(\operatorname{Sym} X,\circ)$, respectively.

$$\begin{aligned}
\phi : (S(\hexagon),\circ) &\longrightarrow (\operatorname{Sym} X,\circ)\\
e &\longmapsto i\\
a &\longmapsto (A\ B\ C)\\
b &\longmapsto (A\ C\ B)\\
c &\longmapsto i\\
d &\longmapsto (A\ B\ C)\\
f &\longmapsto (A\ C\ B)\\
r &\longmapsto (B\ C)\\
s &\longmapsto (A\ B)\\
t &\longmapsto (A\ C)\\
u &\longmapsto (B\ C)\\
v &\longmapsto (A\ B)\\
w &\longmapsto (A\ C)
\end{aligned}$$

(b) The homomorphism ϕ is as follows, where e and i are the identity elements of $(S(\hexagon),\circ)$ and $(\operatorname{Sym} X,\circ)$, respectively.

$$\begin{aligned}
\phi : (S(\hexagon),\circ) &\longrightarrow (\operatorname{Sym} X,\circ)\\
e &\longmapsto i\\
a &\longmapsto (J\ K)(L\ M)\\
b &\longmapsto i\\
c &\longmapsto (J\ K)(L\ M)\\
d &\longmapsto i\\
f &\longmapsto (J\ K)(L\ M)\\
r &\longmapsto (J\ L)(K\ M)\\
s &\longmapsto (J\ M)(K\ L)\\
t &\longmapsto (J\ L)(K\ M)\\
u &\longmapsto (J\ M)(K\ L)\\
v &\longmapsto (J\ L)(K\ M)\\
w &\longmapsto (J\ M)(K\ L)
\end{aligned}$$

Solution to Exercise E190

(a) For the group action in Exercise E189(a), the subsets in the partition of $S(\hexagon)$ and their corresponding permutations of X are as shown below.

Subset in partition of $S(\hexagon)$	Corresponding permutation of X
$\{e,c\}$	i
$\{a,d\}$	$(A\ B\ C)$
$\{b,f\}$	$(A\ C\ B)$
$\{r,u\}$	$(B\ C)$
$\{s,v\}$	$(A\ B)$
$\{t,w\}$	$(A\ C)$

(b) For the group action in Exercise E189(b), the subsets in the partition of $S(\hexagon)$ and their corresponding permutations of X are as shown below.

Subset in partition of $S(\hexagon)$	Corresponding permutation of X
$\{e,b,d\}$	i
$\{a,c,f\}$	$(J\ K)(L\ M)$
$\{r,t,v\}$	$(J\ L)(K\ M)$
$\{s,u,w\}$	$(J\ M)(K\ L)$

(Notice that in each of parts (a) and (b) the listed permutations of X form a subgroup of $\operatorname{Sym} X$, as expected in view of Corollary E72(a).

It also follows from the above and Corollary E72(b) that $\{e,c\}$ and $\{e,b,d\}$ are normal subgroups of $S(\hexagon)$.)

Solution to Exercise E191

We check the group action axioms.

GA1 Let $g \in G$ and let $x \in X$. We have to show that $g \wedge x \in X$. Now

$$\begin{aligned} g \wedge x &= (\phi(g))(x) \quad \text{(by the definition of } \wedge) \\ &\in X \quad \text{(since } \phi(g) \text{ is a permutation of } X). \end{aligned}$$

Thus axiom GA1 holds.

GA2 Let e be the identity element of $(G, *)$ and let $x \in X$. We have to show that $e \wedge x = x$. Let i be the identity element of $(\operatorname{Sym} X, \circ)$; that is, i is the identity permutation of X. Now

$$\begin{aligned} e \wedge x &= (\phi(e))(x) \quad \text{(by the definition of } \wedge) \\ &= i(x) \\ &\quad \text{(since } \phi(e) = i, \text{ as } \phi \text{ is a homomorphism)} \\ &= x. \end{aligned}$$

Thus axiom GA2 holds.

GA3 Let $g, h \in G$ and let $x \in X$. We have to show that $g \wedge (h \wedge x) = (g * h) \wedge x$. Now

$$\begin{aligned} & g \wedge (h \wedge x) \\ &= g \wedge (\phi(h)(x)) \quad \text{(by the definition of } \wedge) \\ &= \phi(g)(\phi(h)(x)) \quad \text{(by the definition of } \wedge) \\ &= (\phi(g) \circ \phi(h))(x) \\ &\qquad \text{(by the definition of function composition)} \\ &= (\phi(g * h))(x) \quad \text{(since } \phi \text{ is a homomorphism)} \\ &= (g * h) \wedge x \quad \text{(by the definition of } \wedge). \end{aligned}$$

Thus axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.